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To Ita, Bracha, Pnina, Pinchas, and Ruth

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Preface

This book is based on lecture notes from a second-year graduate course, and is a greatly expanded version of our previous monograph [K8]. We expose some aspects of the theory of semigroups of linear operators, mostly (but not only) from the point of view of its meeting with that part of spectral theory which is concerned with the integral representation of families of operators. This approach and selection of topics differentiate this book from others in the general area, and reflect the author's own research directions. There is no attempt therefore to cover thoroughly the theory of semigroups of operators. This theory and its applications are extensively exposed in many books, from the classic Hille–Phillips monograph [HP] to the most recent textbook of Engel and Nagel [EN2] (see [A], [BB], [Cl], [D3], [EN1], [EN2], [Fat], [G], [HP], [P], [Vr], and others), as well as in chapters in more general texts on Functional Analysis and the theory of linear operators (cf. [D5], [DS I–III], [Kat1], [RS], [Y], and many others). Nevertheless, because the book is based on a course, and because we intended to make it reasonably self-contained and convenient both for independent study and for a graduate course or seminar, we have included in Section A of Part I (making it thereby the longest section of the book!) an exposition of the basic theory: the classical Hille–Yosida theory on the interplay between a semigroup and its generator up to the characterization of the generator of a (strongly continuous) semigroup by means of estimates on the resolvent iterates, the Lumer–Phillips theory of dissipative operators with its "resolvent-free" characterization of the generator, the Trotter–Kato convergence theorem on the equivalence of "graph convergence" of generators and "strong convergence" of the corresponding semigroups, the Kato unified treatment of the "exponential formula" and the "Trotter product formula," and the Hille–Phillips perturbation theorem for generators of C_o-semigroups. As a transition to the "integral representations" mentioned above, we conclude this section with Stone's theorem for (semi)groups of *unitary* operators and Sz.-Nagy's spectral integral representation for *bounded groups* of operators in Hilbert space.

In Section B of Part I, we construct the *semi-simplicity space* for a given C_o-group of operators in Banach space. It is a Banach subspace which is maximal for the existence of a spectral integral representation of the group on it.

In Section C, we are concerned with *analytic semigroups*, that is, semigroups that possess an analytic continuation to some sector in the complex plane. We present an approach independent of contour integrals, that yields easily to characterizations of the generators of such semigroups.

The semigroup is considered as a *function of its generator* in Section D. We prove a "noncommutative Taylor formula," and consider families of semigroups whose generators depend analytically on a complex parameter in a natural sense. The conceptual meaning of the latter analysis is the hereditary property of analyticity from the coefficients of an Abstract Cauchy Problem to its solution.

The *asymptotic behavior* of (one-parameter) semigroups for large values of the parameter is taken up in Section E. We first consider the relatively simple case of analytic semigroups and of various kinds of "averages" of a semigroup, which include as special cases its Cesaro, Abel, and Gauss averages. We then prove the Arendt–Batty–Lyubich–Vu ("ABLV") stability theorem, using the technique of the so-called "asymptotic space." Adequate conditions on the spectrum of the generator insure the (strong) "stability" of the semigroup, that is, the latter's strong convergence to zero when the parameter tends to infinity. Additional results on stability are included in the "Miscellaneous Exercises" section at the end of the book.

In Section F, we obtain a characterization of generators of *regular semigroups*, that is, analytic semigroups in the right halfplane that possess boundary values on the imaginary axis. We then proceed with the analysis of some classical examples.

A brief discussion of *pre-semigroups*, also called "C-semigroups" or "regularized semigroups" in the literature, concludes Part I of the book (Section G). Pre-semigroups were introduced in germinal form in [DaP], and their extensive study was started in [DP]. They play a role in the solution of the abstract Cauchy problem for an operator which is not necessarily the generator of a semigroup, and is not even densely defined. (The monograph [DL4] presents the theory in great detail, as well as many applications to partial differential equations.)

In Part II, we turn to a more detailed study of integral representations in the spirit of Section B of Part I.

In Section A, the semi-simplicity space is constructed for (generally unbounded) operators *that are not necessarily semigroup generators*, provided they have real spectrum, or at least have a half-line in their resolvent set. A spectral integral representation is obtained for the part of the given operator in its semi-simplicity space, and the latter is a maximal Banach subspace with this property.

In an analogous manner, the *Laplace–Stieltjes space* and the *integrated Laplace space* for a family of closed operators are constructed in Section B by an adequate *renorming method*. As applications, we obtain a spectral integral representation for semigroups of *closed* operators, and a characterization of generators of *n-times integrated semigroups*.

Section C takes up the spectral integral representation for families of unbounded symmetric operators in Hilbert space, defined only *locally* (with respect to the parameter) in a suitable sense. We present the Frohlich–Klein–Landau theory of *local semigroups of unbounded symmetric operators*, generalizing the classical Stone theorem, and an analogous theory for *cosine families of unbounded symmetric operators*. These theories provide a natural approach to Nelson's *Analytic Vectors Theorem* and to Nussbaum's *Semi-analytic Vectors Theorem*, respectively.

Part III contains a small dose of applications, selected from the vast material in the literature by the criterion of our own involvement in their derivations. As mentioned at the beginning of this Introduction, our choice avoids overlapping with the existing monographs dealing with applications of operator semigroup theory in areas such as Markov processes, the Abstract Cauchy Problem, evolution equations, Mathematical Physics, etc. We refer the interested reader to the latter texts, some of which are listed in the Bibliography section.

In Section A of Part III, the results on analytic families of semigroups (exposed in Section D of Part I) are applied to the Abstract Cauchy Problem in the "temporally inhomogeneous" case. Under either Kato's or Tanabe's conditions, it is shown that "coefficients analyticity" implies "solutions analyticity" (with respect to an auxiliary complex parameter).

In Section B, we apply the results of Sections A and F of Part I to the analysis of similarity within the family of operators $S + \zeta V$ (where ζ is a complex parameter), when iS generates a C_o-group $S(\cdot)$, and V is a bounded operator satisfying with S the so-called *Volterra commutation relation* $[S, V] \subset V^2$. This study is motivated by the classical pair of operators on $L^p(0, 1)$, $1 < p < \infty$, defined by $S : f(x) \to xf(x)$ and $V : f(x) \to \int_0^x f(s)\,ds$. In this latter case, $S + \zeta V$ is similar to $S + \omega V$ if and only if $\Re\zeta = \Re\omega$ (cf. [K19]). In the abstract situation (under some additional condition on V), $S + \zeta V$ is similar to S if and only if $\Re\zeta = 0$. Thus, in particular, $S - V$ is *not* similar to S. However, it is proved in the last subsection that the perturbations $(S - V) + P$ *are* similar to S for all P in the "similarity suborbit" $\{S(-t)VS(t);\, t \in \mathbb{R}\}$ of V.

A collection of "exercises" is appended to the main text. In many cases, the exercise contains a significant result, which is reached through the given sequence of steps.

Part I

General Theory

A

Basic Theory

A.1 Overview

The central concept in the theory of operator semigroups is that of the *generator* (or "infinitesimal generator") of the semigroup. In the simplest case of the (semi)group

$$T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \qquad (t \in \mathbb{R})$$

with A a *bounded* (everywhere defined linear) operator, the generator is the operator A. This "generator" clearly contains all the information we might need on the semigroup $T(\cdot)$, and can be retrieved from $T(\cdot)$ by taking the (right) derivative at zero. On the other hand, $T(\cdot)$ can be retrieved from A as the operator solution of the abstract Cauchy initial value problem

$$\frac{dT(t)}{dt} = AT(t); \quad T(0) = I,$$

where I stands for the identity operator. This simple situation occurs for example in the solution of a system of linear ordinary differential equations with constant coefficients (view A as the coefficients matrix, and $T(\cdot)$ as the normalized *fundamental matrix* of the system, cf. [CL]).

In the general situation, an operator semigroup is a function

$$T(\cdot) : [0, \infty) \to B(X)$$

(where $B(X)$ denotes the Banach algebra of all bounded linear operators on the given Banach space X), such that

$$T(s)T(t) = T(s+t) \quad s, t \geq 0$$

(the *semigroup identity*) and $T(0)$ is the identity operator I. In this monograph, we shall consider only semigroups with the additional "C_o-property,"

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called "C_o-semigroups," or semigroups *strongly right-continuous at zero*. The generator A of $T(\cdot)$ is defined as the strong right derivative of $T(\cdot)$ at zero, with maximal domain $D(A)$. The interplay between A and $T(\cdot)$ is the primary subject of this section. We first show that strong (right) continuity at zero implies strong continuity on $[0, \infty)$ and "exponential growth" of the semigroup. We then prove that the generator is a closed densely defined operator. For each $x \in D(A)$, $u = T(\cdot)x$ is the unique C^1 solution of the Abstract Cauchy Problem (ACP)

$$\frac{du}{dt} = Au; \quad u(0) = x.$$

The generator A is bounded (and everywhere defined) if and only if the semigroup $T(\cdot)$ generated by A is continuous in the uniform operator topology on $B(X)$; the semigroup is then of the form $T(t) = e^{tA}$.

We show that the "C^∞-vectors" for the generator A form a (convenient) *core* for it (knowing a core for A is important because it is often difficult to determine the exact domain explicitly).

Another simplified way to study unbounded operators goes through their *resolvent* (which is a *bounded* operator!). In case of a semigroup generator A, the resolvent is well-defined in a right halfplane $\Re\lambda > a$, and is equal there to the Laplace transform of the semigroup. This relation implies a growth estimate on the resolvent iterates, which turns out to be also sufficient for A to generate a C_o-semigroup. This is the famous *Hille–Yosida characterization theorem* for generators. For an arbitrary (unbounded) operator A whose resolvent set contains a half-line, we use a "renorming method" suggested by these estimates to construct a maximal Banach subspace Z of X, such that the *part of A in Z*, A_Z, generates a C_o-semigroup on Z (Z is the so-called *Hille–Yosida space* for A). We then present the Lumer–Phillips resolvent-free characterizations and perturbation invariance results for generators of contraction C_o-semigroups, using the concept of *dissipativity*.

In many applications, A may be approximated in some sense by "simpler" operators. For example, the proof of the Hille–Yosida theorem uses the so-called *Hille–Yosida approximations* of A, which are *bounded* operators converging "pointwise" to A on the latter's domain. In the theory of partial differential operators, an operator with variable coefficients may be approximated (locally) by an operator with constant coefficients, or even by the latter's term of highest order. This motivates the study of the relation between generators convergence (in some sense) and semigroups convergence, and of *perturbations* of semigroup generators. The relevant results are the *Trotter–Kato convergence theorem* and the *Hille–Phillips perturbation theorem*, respectively. Two important consequences of the Trotter–Kato convergence theorem are the *exponential formula* and the *Trotter product formula*. The first formula retrieves $T(\cdot)$ from A in a way analogous to the Euler formula of calculus

$$e^{at} = \lim_n \left(1 - \frac{at}{n}\right)^{-n} = \lim_n \left[\frac{n}{t} R\left(\frac{n}{t}; a\right)\right]^n,$$

where $R(\lambda; a) := (\lambda - a)^{-1}$ is the "resolvent" of the scalar a. The expression on the right with A replacing a, with the limit understood in the strong operator topology, is in fact equal to $T(t)$ for all $t > 0$. The second formula, which has important applications in Mathematical Physics, generalizes the trivial relation

$$e^{t(a+b)} = \lim_n \left[e^{(t/n)a} e^{(t/n)b} \right]^n$$

(for scalars a, b and $c = a + b$) to the case of generators A, B and "$C = A + B$" of generally noncommuting contraction semigroups $S(\cdot)$, $T(\cdot)$, and $U(\cdot)$, respectively (which replace the exponentials in the above scalar formula).

In the last subsection, we prove the classical Stone theorem, to the effect that a C_o-(semi)group of unitary operators $T(\cdot)$ on Hilbert space has the form

$$T(t) = e^{itH},$$

where H is a selfadjoint operator, and the exponential is defined by means of the operational calculus for H provided by the classical Spectral Theorem, that is,

$$T(t) = \int_{\mathbb{R}} e^{its} E(ds),$$

where E is the spectral measure for H (its so-called "resolution of the identity"). This integral representation theorem is valid more generally for any uniformly bounded group of operators in Hilbert space, and various generalizations to a Banach space setting of such integral representations will be elaborated in subsequent sections of Parts I and II.

A.2 The Generator

In this section, we define the generator A of a C_o-semigroup $T(\cdot)$. We prove that $T(\cdot)$ is strongly continuous and $\|T(t)\| \leq Me^{at}$ (with constants $M \geq 1$ and $a \geq 0$) in $[0, \infty)$, that A is a closed operator with dense $T(\cdot)$-invariant domain, $D(A)$, and that for each $x \in D(A)$, the vector function $u = T(\cdot)x$ is strongly differentiable and is the unique C^1 solution of the Abstract Cauchy Problem

$$\frac{du}{dt} = Au; \quad u(0) = x.$$

Let X be a Banach space, and let $B(X)$ denote the Banach algebra of all bounded (linear) operators on X into X.

A function $T(\cdot) : [0, \infty) \to B(X)$ is a semigroup if

$$T(s)T(t) = T(s + t) \qquad (s, t \geq 0) \tag{1}$$

and

$$T(0) = I,$$

where I denotes the identity operator.

The *generator* A of the semigroup $T(\cdot)$ is the operator

$$Ax = \lim_{t \to 0+} t^{-1}[T(t)x - x] \tag{2}$$

with "maximal domain" $D(A)$ consisting of all $x \in X$ for which the limit (2) exists in X (with respect to the norm). This limit is in fact the strong right derivative of $T(\cdot)x$ at 0.

The *continuity at 0* (or C_o) condition is

$$\lim_{t \to 0+} T(t)x = x \tag{3}$$

for all $x \in X$ (limit in X with respect to the norm). This is right continuity at zero in the strong operator topology (s.o.t.) on $B(X)$; in brief, "strong continuity at 0." A C_o-*semigroup* is a semigroup of operators that satisfies the C_o-condition.

Theorem 1.1. *Let $T(\cdot)$ be a C_o-semigroup. Then*

(a) $T(\cdot)$ *is strongly continuous on* $[0, \infty)$, *and*
(b) *there exist constants* $M \geq 1$ *and* $a \geq 0$ *such that*

$$\|T(t)\| \leq M\, e^{at}$$

for all $t \geq 0$.

Proof. Denote

$$c_n = \sup_{0 \leq t \leq 1/n} \|T(t)\| \quad (n \in \mathbb{N}).$$

If $c_n = \infty$ for all n, there exist $t_n \in [0, 1/n]$ such that $\|T(t_n)\| > n$ $(n \in \mathbb{N})$. Then

$$\sup_n \|T(t_n)\| = \infty,$$

and so, by the Uniform Boundedness Theorem, there exists x such that

$$\sup_n \|T(t_n)x\| = \infty.$$

However, the sequence $\|T(t_n)x\|$ converges to $\|x\|$ (by the C_o-condition, since $t_n \to 0+$), and is therefore bounded. This contradiction shows that there exists an n for which $c_n < \infty$. *Fix* such an n, and denote $c = c_n$. Note that $c \geq \|T(0)\| = \|I\| = 1$.

For any $t > 0$, the semigroup property gives

$$T(t) = T(1/n)^{n[t]} T(\{t\}/n)^n,$$

where $[t]$ denotes the *entire part* of t, and $\{t\}$ its *fractional part*. Since $1/n$ and $\{t\}/n$ are both in $[0, 1/n]$, we have $\|T(1/n)\| \leq c$ and $\|T(\{t\}/n)\| \leq c$, so that

$$\|T(t)\| \leq (c^n)^{[t]+1} \leq (c^n)^{t+1} = Me^{at},$$

where $M = c^n \geq 1$ and $a = n \log c \geq 0$ (we used the fact that $c \geq 1$).

For $h > 0$, we have for all $x \in X$

$$\|T(t+h)x - T(t)x\| = \|T(h)[T(t)x] - [T(t)x]\| \to 0$$

as $h \to 0$, by the C_o condition with the fixed vector $T(t)x$.

For $h < 0$, write $h = -k$, with $0 < k < t$. Then

$$\begin{aligned}\|T(t+h)x - T(t)x\| &= \|T(t-k)(x - T(k)x)\| \\ &\leq Me^{a(t-k)}\|T(k)x - x\| \to 0\end{aligned}$$

as $h \to 0$ by the C_o property. □

Theorem 1.2. *Let A be the generator of the C_o-semigroup $T(\cdot)$. Then:*

1. *A is closed and densely defined.*
2. *For each $t \geq 0$, $T(t)D(A) \subset D(A)$, and*

$$AT(t)x = T(t)Ax = \frac{d}{dt}[T(t)x]$$

for each $x \in D(A)$.

3. *For each $x \in D(A)$, the function $u = T(\cdot)x$ is C^1 on $[0, \infty)$, and is the unique solution of the Abstract Cauchy Problem (ACP) on $[0, \infty)$:*

$$\frac{du}{dt} = Au; \quad u(0) = x. \tag{ACP}$$

Proof. For each given $x \in X$, the function $T(\cdot)x$ is continuous on $[0, \infty)$, by Theorem 1.1, and has therefore a Riemann integral over any finite interval $[0, t]$. Denote this integral by x_t. Also let $A_h = h^{-1}[T(h) - I]$ for $h > 0$. Then

$$\begin{aligned}A_h x_t &= h^{-1}\left[\int_0^t T(s+h)x\,ds - \int_0^t T(s)x\,ds\right] \\ &= h^{-1}\left[\left(\int_h^{t+h} - \int_0^t\right) T(s)x\,ds\right] \\ &= h^{-1}\int_t^{t+h} T(s)x\,ds - h^{-1}\int_0^h T(s)x\,ds \\ &\to T(t)x - x\end{aligned}$$

as $h \to 0+$, by continuity of $T(\cdot)x$.

Hence $x_t \in D(A)$ and

$$Ax_t = T(t)x - x. \tag{4}$$

The C_o-condition implies that $x_t/t (\in D(A)) \to x$, and therefore $D(A)$ is dense in X.

If $x \in D(A)$, then for each $t > 0$,

$$A_h T(t)x = T(t)A_h x \to T(t)Ax \tag{5}$$

as $h \to 0+$. Hence $T(t)x \in D(A)$ and

$$AT(t)x = T(t)Ax.$$

The left-hand side in (5) is also equal to

$$h^{-1}[T(t+h)x - T(t)x],$$

and so the right derivative of $T(\cdot)x$ exists, is equal to $A[T(\cdot)x] = T(\cdot)(Ax)$, and is in particular continuous.

If $0 < k < t$, we have for $x \in D(A)$

$$\begin{aligned} &\|(-k)^{-1}[T(t-k)x - T(t)x] - T(t)Ax\| \\ &\quad \le \|T(t-k)(A_k x - Ax)\| + \|T(t-k)Ax - T(t)Ax\| \\ &\quad \le Me^{a(t-k)}\|A_k x - Ax\| + \|T(t-k)Ax - T(t)Ax\| \to 0 \end{aligned}$$

as $k \to 0+$, since $x \in D(A)$ and $T(\cdot)$ is strongly continuous.

Thus $u = T(\cdot)x$ is of class C^1 on $[0, \infty)$, and solves (ACP).

Suppose $v : [0, \infty) \to D(A)$ is differentiable. Then

$$\begin{aligned} &h^{-1}[T(t+h)v(t+h) - T(t)v(t)] \\ &\quad = T(t)A_h v(t) + T(t+h)[h^{-1}(v(t+h) - v(t)) - v'(t)] + T(t+h)v'(t). \end{aligned}$$

The first term on the right has limit $T(t)Av(t)$ when $h \to 0$, since $v(t) \in D(A)$ and $T(t) \in B(X)$. The second term has limit 0, since $\|T(t+h)\| \le Me^{a(t+h)}$. The last term has limit $T(t)v'(t)$, by strong continuity of $T(\cdot)$. Hence

$$\frac{d}{dt}[T(t)v(t)] = T(t)[Av(t) + v'(t)].$$

Suppose now that v solves (ACP) with a given $x \in D(A)$, in some interval $[0, \tau]$. Fix $s \in (0, \tau]$. Then by the fundamental theorem of calculus

$$\begin{aligned} T(s)x - v(s) &= \int_0^s \frac{d}{dt}[T(t)v(s-t)]\,dt \\ &= \int_0^s T(t)[Av(s-t) - v'(s-t)]\,dt = 0, \end{aligned}$$

which proves the uniqueness.

By the fundamental theorem of calculus and (4),

$$A\int_0^t T(s)x\,ds = T(t)x - x = \int_0^t T(s)Ax\,ds$$

for $t > 0$ and $x \in D(A)$.

Suppose $x_n \in D(A)$ are such that $x_n \to x$ and $Ax_n \to y$ in X. If $V(\cdot) : [0,\tau] \to B(X)$ is strongly continuous, then $\|V(\cdot)\|$ is a bounded measurable function and for each $x \in X$,

$$\left\| \int_0^\tau V(t)x\, dt \right\| \le \int_0^\tau \|V(t)\|\, dt\, \|x\|$$

(see below).

Therefore, as $n \to \infty$,

$$\begin{aligned} &\left\| \int_0^t T(s)Ax_n\, ds - \int_0^t T(s)y\, ds \right\| \\ &\quad \le \int_0^t \|T(s)\|\, ds\, \|Ax_n - y\| \to 0. \end{aligned}$$

Hence

$$\begin{aligned} A_t x &= \lim_n A_t x_n = \lim_n t^{-1} \int_0^t T(s)Ax_n\, ds \\ &= t^{-1} \int_0^t T(s)y\, ds. \end{aligned}$$

It follows that $\lim_{t\to 0+} A_t x = y$.

This proves that $x \in D(A)$ and $Ax = y$, i.e., A is closed.

Back to the claim about $V(\cdot)$, the boundedness of $\|V(\cdot)\|$ follows immediately from the strong continuity of $V(\cdot)$ and the Uniform Boundedness Theorem. To prove the measurability of $\|V(\cdot)\|$, it suffices to show that the set $C = \{t \in [0,\tau]; \|V(t)\| > c\}$ is Borel for each $c \ge 0$. If $t \in C$, there exists $x \in X$ with norm 1 such that $\|V(t)x\| > c$, and by continuity of $\|V(\cdot)x\|$, there is a neighborhood of t in $[0,\tau]$ where $\|V(\cdot)x\| > c$, and so $\|V(\cdot)\| > c$ there. Hence C is open, so certainly Borel. (We proved actually that $\|V(\cdot)\|$ is lower semicontinuous.) □

A.3 Type and Spectrum

We shall define the *type* ω of the C_o-semigroup $T(\cdot)$, and show that it is related to the spectral radius $r(T(t))$ of $T(t)$ by the formula $r(T(t)) = e^{\omega t}$ $(t \ge 0)$.

By Theorem 1.1, $\log \|T(\cdot)\|$ is bounded above on finite intervals and clearly sub-additive. We need the following general lemma on such functions.

Lemma 1.3. *Let* $p : [0,\infty) \to [-\infty,\infty)$ *be sub-additive (i.e.,* $p(t+s) \le p(t) + p(s)$ *for all* $t, s \ge 0$*) and bounded above in* $[0,1]$*. Then*

$$-\infty \le \inf_{t>0} \frac{p(t)}{t} = \lim_{t\to\infty} \frac{p(t)}{t} < \infty.$$

Proof. If $p(t_0) = -\infty$ for some t_0, then for all $t \geq t_0$, $p(t) \leq p(t_0) + p(t - t_0) = -\infty$, and the result is trivial. So we may assume that p is finite. Fix $s > 0$ and $r > p(s)/s$. For $t > 0$ arbitrary, let n be the unique positive integer such that $ns \leq t < (n+1)s$. Then

$$\begin{aligned} p(t)/t &= p(ns + (t - ns))/t \leq np(s)/t + p(t - ns)/t \\ &< rns/t + \sup_{[0,s]} p/t. \end{aligned}$$

Since the hypothesis implies that p is bounded above on any interval $[0, s]$, it follows that $\limsup_{t\to\infty} \frac{p(t)}{t} \leq r$, for any $r > p(s)/s$. Hence

$$\limsup \frac{p(t)}{t} \leq \inf_{s>0} \frac{p(s)}{s} \leq \liminf \frac{p(t)}{t},$$

and the lemma follows. □

In particular, the *type* of $T(\cdot)$ is (fixed notation!)

$$\omega := \inf_{t>0} \frac{\log \|T(t)\|}{t} = \lim_{t\to\infty} \frac{\log \|T(t)\|}{t}.$$

For any non-negative $a > \omega$, we clearly have

$$\|T(t)\| \leq Me^{at}$$

for all $t \geq 0$ (where the constant $M \geq 1$ depends on a).

Theorem 1.4. *The spectral radius of* $T(t)$ *is* $e^{\omega t}$.

Proof. Since the claim is trivial for $t = 0$, fix $t > 0$, and let $r(T(t))$ denote the spectral radius of $T(t)$. By the Beurling–Gelfand formula and Lemma 1.3, we have

$$\begin{aligned} r(T(t)) &= \lim_n \|T(t)^n\|^{1/n} = \lim_n e^{(1/n)\log\|T(nt)\|} \\ &= e^{t \lim_n (1/nt)\log\|T(nt)\|} = e^{\omega t}. \end{aligned}$$

□

A.4 Uniform Continuity

The next theorem shows that the stronger hypothesis of right continuity at zero in the *uniform operator topology* (that is, in the norm topology of $B(X)$) is equivalent to the condition $A \in B(X)$, and yields to the obvious class of semigroups of the form

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

Theorem 1.5. *The semigroup $T(\cdot)$ is right continuous at 0 in the uniform operator topology if and only if its generator A belongs to $B(X)$. In that case, $T(t) = e^{tA}$, where the exponential is defined by the usual power series, which converges in $B(X)$.*

Proof. If $A \in B(X)$, one verifies directly that e^{tA} is a well-defined norm-continuous group with generator A. Since by Theorem 1.2 the generator determines the semigroup uniquely, and A is also the generator of $T(\cdot)$, we have $T(t) = e^{tA}$, so that, in particular, $T(\cdot)$ is norm-continuous.

Suppose conversely that $T(\cdot)$ is norm-continuous at 0 (hence everywhere on $[0, \infty)$, by the argument in the proof of Theorem 1.1). We may then consider Riemann integrals of $T(\cdot)$, defined as the usual limits (in $B(X)$!). For $h, t > 0$, a calculation as at the beginning of the proof of Theorem 1.2 shows that

$$[T(t) - I] \int_0^h T(s)\, ds = [T(h) - I] \int_0^t T(s)\, ds.$$

Since

$$\lim_{h \to 0+} \left\| h^{-1} \int_0^h T(s)\, ds - I \right\| = 0$$

by norm-continuity of $T(\cdot)$ at 0, we can *fix* h such that the above norm is less than 1, and therefore $V := \int_0^h T(s)\, ds$ is invertible in $B(X)$. Hence,

$$T(t) - I = \left(\int_0^t T(s)\, ds \right) A,$$

where

$$A := [T(h) - I]V^{-1} \in B(X).$$

(The change of order in the calculation is valid, since the values of $T(\cdot)$ commute.) Dividing by t and letting $t \to 0$, we get $t^{-1}[T(t) - I] \to A$ in $B(X)$, by norm-continuity of $T(\cdot)$. Hence $A (\in B(X)!)$ is the generator of $T(\cdot)$. □

A.5 Core for the Generator

In practice, it is not necessary to determine exactly the domain $D(A)$, and it suffices to know a *core* for A. We show that the C^∞*-vectors* form a core for A.

Definition 1.6. *Let A be any closed operator with domain $D(A)$ in X. The graph-norm on $D(A)$ is the norm*

$$|x|_A := \|x\| + \|Ax\|$$

induced on the graph of A by the norm on X^2.

$D(A)$ is a Banach space under the graph-norm (because A is closed), and we shall use the notation $[D(A)]$ for this Banach space. Any subspace D_0 dense in $[D(A)]$ is called a *core* for A. Explicitly, a subspace D_0 of $D(A)$ is a core for A iff for any $x \in D(A)$, there exists a sequence $\{x_n\}$ in D_0 such that $x_n \to x$ and $Ax_n \to Ax$ (i.e., A equals the closure $\overline{A|_{D_0}}$ of its restriction to D_0).

The following theorem gives a useful sufficient condition for a subspace D_0 to be a core for the *generator* A of a semigroup $T(\cdot)$.

Theorem 1.7. *Let A be the generator of the C_o-semigroup $T(\cdot)$. If D_0 is a subspace of $D(A)$ dense in X and $T(\cdot)$-invariant, then it is a core for A.*

Proof. Note first that $T(\cdot)$ is a C_o-semigroup in the Banach space $[D(A)]$, since for all $x \in D(A)$, when $t \to 0+$,

$$|T(t)x - x|_A = \|T(t)x - x\| + \|T(t)(Ax) - (Ax)\| \to 0.$$

Therefore, for $x \in D_0$, Riemann integrals (over finite intervals) of $T(\cdot)x$ make sense in the graph-norm, and belong to $\overline{D_0}$, the closure of D_0 in $[D(A)]$. Let $x \in D(A)$. By density of D_0 in X, there exists a sequence $\{x_n\} \subset D_0$ such that $x_n \to x$ in X. The elements x_t and $(x_n)_t$ (see notation in proof of Theorem 1.2) are in $D(A)$, and for each $t > 0$

$$|(x_n)_t - x_t|_A = \left\| \int_0^t T(s)(x_n - x)\, ds \right\| + \|[T(t)x_n - x_n] - [T(t)x - x]\| \to 0$$

when $n \to \infty$. Since $(x_n)_t \in \overline{D_0}$ for each n, we have also $x_t \in \overline{D_0}$. Finally, by the C_o-property of $T(\cdot)$ in $[D(A)]$, $t^{-1}x_t (\in \overline{D_0}!) \to x$ in the graph-norm, and so $x \in \overline{D_0}$. □

A useful core for A is the space $D^\infty = D^\infty(A)$ of all C^∞*-vectors* for A, that is, the set of all $x \in X$ for which the vector function $T(\cdot)x$ is of class C^∞ (in the strong sense) on $[0, \infty)$.

Theorem 1.8.

1. $D^\infty = \bigcap_{n=1}^\infty D(A^n)$.
2. D^∞ *is dense in X and $T(\cdot)$-invariant.*
3. D^∞ *is a core for A.*

Proof. 1 and 2 imply 3 by Theorem 1.7.

If $x \in D^\infty$, $T(\cdot)x$ is differentiable at 0, i.e., $x \in D(A)$, and $(d/dt)T(t)x = T(t)(Ax)$. Hence $Ax \in D^\infty$, and so, in particular, $x \in D(A^2)$. Inductively, $x \in D(A^n)$ and

$$[T(\cdot)x]^{(n)} = T(\cdot)A^n x \tag{1}$$

for all $n = 1, 2, 3 \ldots$.

Conversely, if $x \in D(A^n)$ for all n, then $T(\cdot)x$ is differentiable and $[T(\cdot)x]' = T(\cdot)Ax$ (cf. Theorem 1.2), so that, inductively, we obtain that $T(\cdot)x$ is of class C^∞ and (1) is valid. This proves 1 and the $T(\cdot)$-invariance of D^∞.

To prove the density of D^∞, we use an "approximate identity" $0 \le h_n \in C^\infty(\mathbb{R})$ with support in $(0, 1/n)$ and integral (over $\mathbb{R}$) equal to 1. Given $x \in X$, define

$$x_n = \int_0^\infty h_n(t)T(t)x\,dt.$$

Then $x_n \to x$ in X, and it remains to show that $x_n \in D^\infty$ for all n. For $k > 0$,

$$\begin{aligned} A_k x_n &= k^{-1} \int_0^\infty h_n(t)[T(t+k)x - T(t)x]\,dt \\ &= \int_0^\infty k^{-1}[h_n(t-k) - h_n(t)]T(t)x\,dt \\ &\to -\int_0^\infty h_n'(t)T(t)x\,dt \end{aligned}$$

when $k \to 0+$. Hence $x_n \in D(A)$ and $Ax_n = -\int_0^\infty h_n'(t)T(t)x\,dt$. Repeating the argument, we obtain $x_n \in D(A^j)$ for all j and $A^j x_n = (-1)^j \int_0^\infty h_n^{(j)}(t)\,T(t)x\,dt$. The conclusion follows now from 1. □

A.6 The Resolvent

When we deal with an unbounded operator A, it is convenient to study it by means of the operator $R(\lambda; A) := (\lambda I - A)^{-1}$, if the latter does exist as a *bounded everywhere defined operator* for scalars λ in a sufficiently "large" subset of the complex plane. The basic properties of the $B(X)$-valued function $R(\cdot; A)$ (called the *resolvent* of A) are the subject of this section.

The verification of the following elementary facts is left as an exercise.

Proposition 1.9. *Let A be a closed operator, with domain $D(A)$. Then:*

1. *If A is injective, its inverse with domain $D(A^{-1})$ equal to the range* $\mathrm{ran}(A)$ *of A, is closed.*
2. *If $B \in B(X)$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha A + \beta B$, with domain X for $\alpha = 0$ and $D(A)$ otherwise, is closed.*
3. *If $B \in B(X)$, then AB, with its maximal domain, is closed. If B is non-singular, then BA, with domain $D(A)$, is also closed.*

Definition 1.10. *The resolvent set $\rho(A)$ of the closed operator A is the set of all complex λ for which $\lambda I - A$ is bijective (i.e., one-to-one and onto X). Its complement is the spectrum $\sigma(A)$ of A.*

The operator $R(\lambda) = R(\lambda; A) := (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$ is closed (see 1.9) and everywhere defined, and belongs therefore to $B(X)$ by the closed graph theorem. It is called the *resolvent of* A.

Observe that $\lambda \in \rho(A)$ iff there exists an operator $R(\lambda) \in B(X)$ with range in $D(A)$ such that

$$(\lambda I - A)R(\lambda)x = x \quad (x \in X) \tag{1}$$

and

$$R(\lambda)(\lambda I - A)x = x \qquad (x \in D(A)). \tag{2}$$

It is useful to write the above relations in the form

$$R(\lambda)A \subset AR(\lambda) = \lambda R(\lambda) - I \tag{3}$$

(where all operators are with their maximal domain).

Theorem 1.11. *Let A be a closed operator. Then $\rho(A)$ is open, $R(\cdot)$ is analytic on $\rho(A)$ and satisfies the "resolvent equation"*

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu). \tag{4}$$

Also

$$\|R(\lambda)\| \geq \frac{1}{d(\lambda, \sigma(A))} \tag{5}$$

for all $\lambda \in \rho(A)$, where $d(\cdots)$ denotes the distance from λ to $\sigma(A)$.

Proof. Let $\lambda \in \rho(A)$, and set

$$\delta = \|R(\lambda)\|^{-1}.$$

The series

$$S(\zeta) = \sum_{n \geq 0} (-1)^n R(\lambda)^{n+1} (\zeta - \lambda)^n$$

is norm-convergent in $B(X)$ for $|\zeta - \lambda| < \delta$, and so defines an element of $B(X)$.

For $x \in D(A)$,

$$\begin{aligned} S(\zeta)(\zeta I - A)x &= S(\zeta)[(\zeta - \lambda)I + (\lambda I - A)]x \\ &= \sum (-1)^n R(\lambda)^{n+1} (\zeta - \lambda)^{n+1} x \\ &\quad + \sum (-1)^n R(\lambda)^n (\zeta - \lambda)^n x = x. \end{aligned}$$

Next, for any $x \in X$, let x_m denote the m-th partial sum of the series $S(\zeta)x$. Then $x_m \in D(A)$ (because $x_m \in \mathrm{ran} R(\lambda) = D(A)$), $x_m \to S(\zeta)x$, and by (3)

$$
\begin{aligned}
Ax_m &= \sum_{0\le n\le m} (-1)^n \lambda R(\lambda)^{n+1}(\zeta-\lambda)^n x \\
&\quad - \sum_{0\le n\le m} (-1)^n R(\lambda)^n(\zeta-\lambda)^n x \\
&\to \lambda S(\zeta)x + (\zeta-\lambda)S(\zeta)x - x \\
&= \zeta S(\zeta)x - x.
\end{aligned}
$$

Since A is closed, it follows that $S(\zeta)x \in D(A)$ and $(\zeta I - A)S(\zeta)x = x$, and we conclude that $\zeta \in \rho(A)$ and $R(\zeta) = S(\zeta)$ for all ζ such that $|\zeta - \lambda| < \delta$. Hence $\rho(A)$ is open and $R(\cdot)$ is analytic on $\rho(A)$. Also, since the disk of radius δ around λ is contained in $\rho(A)$, we have

$$d(\lambda, \sigma(A)) \ge \delta := \|R(\lambda)\|^{-1}.$$

Finally, for $\lambda, \mu \in \rho(A)$ and $x \in X$,

$$
\begin{aligned}
&(\lambda I - A)[R(\lambda) - R(\mu) - (\mu-\lambda)R(\lambda)R(\mu)]x \\
&\quad = x - [(\lambda-\mu)I + (\mu I - A)]R(\mu)x - (\mu-\lambda)R(\mu)x \\
&\quad = x - (\lambda-\mu)R(\mu)x - x + (\lambda-\mu)R(\mu)x = 0.
\end{aligned}
$$

Since $\lambda I - A$ is injective, (4) follows. □

A.7 Pseudo-Resolvents

It will be convenient to consider the following general concept of a "pseudo-resolvent" (motivated by the resolvent equation), and to find sufficient condition for a pseudo-resolvent to be in fact the resolvent of some closed operator.

Definition 1.12. *A pseudo-resolvent is a function $R(\cdot)$, defined on an open set $U \subset \mathbb{C}$, with values in $B(X)$, and satisfying the resolvent equation in U.*

Theorem 1.13. *If $R(\cdot) : U \to B(X)$ is a pseudo-resolvent, then* $\mathrm{ker}R(\lambda)$ *and* $\mathrm{ran}R(\lambda)$ *are independent of $\lambda \in U$, and $R(\cdot)$ is the resolvent of some closed operator A with $U \subset \rho(A)$ iff* $\mathrm{ker}R(\lambda) = \{0\}$.

Proof. Let $\lambda, \mu \in \rho(A)$. If $x \in \mathrm{ker}R(\lambda)$, we have by the resolvent equation

$$R(\mu)x = R(\lambda)x + (\lambda-\mu)R(\mu)R(\lambda)x = 0,$$

i.e., $x \in \mathrm{ker}R(\mu)$, and so, by symmetry, $\mathrm{ker}R(\lambda) = \mathrm{ker}R(\mu)$.

If $y \in \mathrm{ran}R(\lambda)$, write $y = R(\lambda)x$, and then

$$y = R(\mu)[x + (\mu-\lambda)R(\lambda)x] \in \mathrm{ran}R(\mu),$$

so that $\mathrm{ran}R(\lambda) = \mathrm{ran}R(\mu)$ by symmetry.

Suppose $\ker R(\lambda) = \{0\}$ for some (hence for all) $\lambda \in U$. Then

$$A := \lambda I - R(\lambda)^{-1} : \operatorname{ran} R(\lambda) \to X$$

is closed, and since $\lambda I - A = R(\lambda)^{-1}$ and $R(\lambda) \in B(X)$, the operator $\lambda I - A$ is bijective. Thus $\lambda \in \rho(A)$ and $R(\lambda) = (\lambda I - A)^{-1}$.

For any $\mu \in U$, we have by the resolvent equation

$$R(\mu) = R(\lambda)[I + (\lambda - \mu)R(\mu)],$$

and therefore

$$(\mu I - A)R(\mu) = (\mu - \lambda)R(\mu) + (\lambda I - A)R(\lambda)[I + (\lambda - \mu)R(\mu)] = I,$$

and similarly $R(\mu)(\mu I - A) \subset I$. Therefore $\mu \in \rho(A)$ and $R(\mu; A) = R(\mu)$.

Conversely, if $R(\lambda) = R(\lambda; A)$ for all $\lambda \in U$ (for some closed operator A), then $R(\cdot)$ is a pseudo-resolvent by Theorem 1.13, $U \subset \rho(A)$, and $\ker R(\lambda) = 0$ trivially. □

Another characterization of resolvents among pseudo-resolvents uses the *range* of $R(\lambda)$ (rather than its kernel).

Theorem 1.14. *Let $R(\cdot) : (\omega, \infty) \to B(X)$ be a pseudo-resolvent such that for all $\lambda > \omega$,*

$$M := \sup_{\lambda > \omega} (\lambda - \omega)\|R(\lambda)\| < \infty. \tag{1}$$

Then $R(\cdot)$ is the resolvent of a closed densely-defined operator iff the range of $R(\lambda)$ is dense in X for some (hence for all) λ.

Proof. The necessity is trivial, since $\operatorname{ran} R(\lambda; A) = D(A)$.

Sufficiency. For $x \in \operatorname{ran} R(\lambda)$, write $x = R(\lambda)y$, and then, for $\mu > \omega$,

$$\begin{aligned}\mu R(\mu)x &= \mu R(\mu)R(\lambda)y \\ &= \frac{\mu}{\mu - \lambda}[R(\lambda)y - R(\mu)y] \\ &\to R(\lambda)y = x\end{aligned}$$

when $\mu \to \infty$, by Condition (1). Since $\operatorname{ran} R(\lambda)$ is dense in X, it follows from (1) that $\mu R(\mu)x \to x$ for all $x \in X$

[indeed, let $x_n \in \operatorname{ran} R(\lambda)$ converge to x. Then

$$\|\mu R(\mu)x - x\| \le \|\mu R(\mu)x_n - x_n\| + \|\mu R(\mu) - I\| \, \|x - x_n\|.$$

The second term on the right-hand side is

$$\le [M\mu/(\mu - \omega) + 1]\|x - x_n\| \to (M + 1)\|x - x_n\|,$$

and the first term $\to 0$ when $\mu \to \infty$ (for each fixed n). Therefore

$$\limsup_{\mu\to\infty} \|\mu R(\mu)x - x\| \leq (M+1)\|x - x_n\|$$

for each n. Letting $n \to \infty$, the conclusion follows].

Suppose $x \in \ker R(\lambda)$ for some $\lambda > \omega$. Then $x \in \ker R(\mu)$ for all $\mu > \omega$, but then $x = \lim \mu R(\mu)x = 0$, i.e., $\ker R(\lambda) = \{0\}$, and so $R(\lambda) = R(\lambda; A)$ for some closed operator A (by Theorem 1.13), and $D(A) = \operatorname{ran} R(\lambda)$ is dense, by hypothesis. □

A.8 The Laplace Transform

We show next that the resolvent of the generator A of the C_o-semigroup $T(\cdot)$ exists in the halfplane $\omega + \mathbb{C}^+$ (where $\mathbb{C}^+$ denotes the open right half-plane), and is equal there to the Laplace transform of $T(\cdot)$. This will imply in particular a growth estimate on the iterates of the resolvent, which will turn out to characterize generators of C_o-semigroups.

Theorem 1.15. *Let $T(\cdot)$ be a C_o-semigroup. Let A be its generator, and ω its type. Then*

1. $\sigma(A) \subset \{\lambda \in \mathbb{C}; \Re\lambda \leq \omega\}$.
2. *For $\Re\lambda > \omega$ and $x \in X$,*

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt. \tag{1}$$

3. *For $c > \omega, t > 0$ and $x \in D(A)$,*

$$T(t)x = \lim_{\tau\to\infty} \frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} e^{\lambda t} R(\lambda; A)x \, d\lambda, \tag{2}$$

where the limit is a strong limit in X.

Proof. For any $a > \omega$, $\|T(t)\| = O(e^{at})$, and therefore the Laplace integral $L(\lambda)x$ defined by the right-hand side of (1) converges absolutely for $\Re\lambda > a$, and defines an operator $L(\lambda) \in B(X)$ satisfying

$$\|L(\lambda)\| \leq \frac{M}{\Re\lambda - a} \tag{3}$$

for a suitable constant M (which depends on a). If $x \in D(A)$,

$$\begin{aligned} L(\lambda)(\lambda I - A)x &= \int_0^\infty \{\lambda e^{-\lambda t} T(t)x - e^{-\lambda t}[T(t)x]'\} \, dt \\ &= -\int_0^\infty [e^{-\lambda t} T(t)x]' \, dt = x. \end{aligned}$$

On the other hand, for any $x \in X$ and $h > 0$,

$$A_h L(\lambda)x = h^{-1} \int_0^\infty e^{-\lambda t}[T(t+h)x - T(t)x]\,dt$$

$$= h^{-1}(e^{\lambda h} - 1)L(\lambda)x - e^{\lambda h} h^{-1} \int_0^h e^{-\lambda t} T(t)x\,ds$$

$$\to_{h \to 0+} \lambda L(\lambda)x - x.$$

Hence $L(\lambda)X \subset D(A)$ and $(\lambda I - A)L(\lambda)x = x$ for all $x \in X$.

We conclude that $L(\lambda) = R(\lambda; A)$ for all λ in the half-plane $\Re\lambda > a$. Since $a > \omega$ was arbitrary, Statements 1 and 2 are proved.

To obtain 3, we observe that $T(\cdot)x$ is of class C^1 on $[0, \infty)$ (by Theorem 1.2), and we may therefore apply the (vector version of the) classical Complex Inversion Theorem for the Laplace transform (cf. Theorem 7.3 in [W]). □

A.9 Abstract Potentials

The Laplace integral representation of $R(\lambda; A)$ implies the growth condition

$$\|R(\lambda; A)\| \le \frac{M}{\lambda - a} \tag{1}$$

for all $\lambda > a$ (where $a > \omega$ is fixed). Consider now *any* densely defined operator A with $(a, \infty) \subset \rho(A)$, which satisfies (1) for all $\lambda > \lambda_0$ (for some $\lambda_0 \ge a$). For short, call such an operator an *abstract potential*. Note that an abstract potential is necessarily closed, since its resolvent set is nonempty.

It is interesting to observe that if A satisfies (1) and $\rho(A)$ contains some sector

$$S_{a,\theta} := \{\lambda = t + is \in \mathbb{C};\ t > a, \quad |s| < (t-a)\tan\theta\}$$

with $0 < \theta < \arctan(1/M)$, then one has

$$\|(\Re\lambda - a)R(\lambda; A)\| \le M',$$

with $M' := M(1 - M\tan\theta)^{-1}$, for *all* $\lambda \in S_{a,\theta}$. (Note that M' is well defined and $> M$.)

Indeed, *fix* $t + is \in S_{a,\theta}$. By (1)

$$\| - isR(t; A)\| \le \tan\theta\, \|(t-a)R(t; A)\| \le M\ \tan\theta < 1.$$

Hence $I + is\,R(t; A)$ is invertible in $B(X)$, and

$$\|[I + is\,R(t; A)]^{-1}\| = \left\| \sum_{k=0}^{\infty} [-is\,R(t; A)]^k \right\|$$

$$\le \sum_k \| - is\,R(t; A)\|^k \le (1 - M\ \tan\theta)^{-1} = M'/M.$$

By the First Resolvent Equation,

$$R(t+is;A)\,[I+isR(t;A)] = R(t;A),$$

that is,

$$R(t+is;A) = R(t;A)\,[I+is\,R(t;A)]^{-1},$$

and therefore by (1) and the preceding estimate,

$$\|(t-a)\,R(t+is;A)\| \le \|(t-a)\,R(t;A)\|\,\|[I+is\,R(t;A)]^{-1}\| \le M'.$$

Lemma 1.16. *Let A be an abstract potential, and consider the bounded operators*

$$A_\lambda := \lambda A R(\lambda) = \lambda[\lambda R(\lambda) - I]$$

for $\lambda > a$, where $R(\cdot) := R(\cdot;A)$. Then as $\lambda \to \infty$,

(a) $A_\lambda x \to Ax$ *for all $x \in D(A)$;*
(b) $\lambda R(\lambda) \to I$ *strongly (equivalently, $AR(\lambda) \to 0$ strongly).*

Proof. For $x \in D(A)$ and $\lambda > \lambda_0$,

$$\|AR(\lambda)x\| = \|R(\lambda)Ax\| \le \frac{M}{\lambda - a}\|Ax\| \to 0.$$

Since

$$\|AR(\lambda)\| = \|\lambda R(\lambda) - I\| \le \frac{\lambda M}{\lambda - a} + 1 = O(1)$$

when $\lambda \to \infty$, and since $D(A)$ is dense in X, it follows that

$$AR(\lambda)x \to 0$$

for all $x \in X$. This is equivalent to (b).

Next, for $x \in D(A)$,

$$A_\lambda x = \lambda R(\lambda)(Ax) \to Ax$$

by (b). □

(Note that the notation A_λ in the present context should not be confused with the notation A_h used in previous sections.)

When A is the generator of a semigroup $T(\cdot)$, and the constants $M \ge 1$ and $a \ge 0$ are chosen such that $\|T(t)\| \le Me^{at}$ (cf. Theorem 1.1), the growth property (1) can be strengthened as follows:

For any finite set of $\lambda_k > a$, $\quad k = 1, \ldots, m$,

$$\left\| \prod_k (\lambda_k - a) R(\lambda_k; A) \right\| \le M. \tag{2}$$

In particular (with all λ_k equal to λ),

$$\|R(\lambda;A)^m\| \leq \frac{M}{(\lambda-a)^m} \tag{3}$$

for all $\lambda > a$ and $m = 1, 2, 3, \ldots$.

Indeed, for all $x \in X$,

$$\begin{aligned}
&\left\|\prod_k (\lambda_k - a)R(\lambda_k;A)x\right\| \\
&\quad = \left\|\int_0^\infty \cdots \int_0^\infty \prod_k (\lambda_k - a)e^{-\lambda_1 t_1 - \cdots - \lambda_m t_m} T(t_1 + \cdots + t_m)x\, dt_1 \cdots dt_m\right\| \\
&\quad \leq M \int_0^\infty \cdots \int_0^\infty \prod_k (\lambda_k - a)e^{-(\lambda_k - a)t_k}\, dt_1 \cdots dt_m \,\|x\| \\
&\quad = M \prod_k \int_0^\infty (\lambda_k - a)e^{-(\lambda_k - a)t}\, dt\, \|x\| = M\,\|x\|.
\end{aligned}$$

A.10 The Hille–Yosida Theorem

We prove now that the iterate resolvent estimates (3) in Subsection A.9 characterize generators of C_o-semigroups among all abstract potentials.

Theorem 1.17 (Hille–Yosida Theorem). *An operator A is the generator of a C_o-semigroup $T(\cdot)$ (satisfying $\|T(t)\| \leq Me^{at}$ for some constants $a \geq 0$ and $M \geq 1$, for all $t \geq 0$) iff*

(a) *A is densely defined; and*
(b) *$\rho(A)$ contains some ray (a, ∞) $(a \geq 0)$ and there exists a finite positive constant M such that*

$$\|R(\lambda;A)^m\| \leq \frac{M}{(\lambda-a)^m}$$

for all $\lambda > a$ and $m \in \mathbb{N}$.

Proof. We already saw the necessity of (a) and (b).

Sufficiency. Let A satisfy (a) and (b). In particular, A is an abstract potential, and so Lemma 1.16 is satisfied. Define

$$T_\lambda(t) = e^{tA_\lambda}.$$

We have for $\lambda > 2a$ (so that $\frac{a\lambda}{\lambda-a} < 2a$):

$$\|T_\lambda(t)\| \leq e^{-\lambda t} \sum_n \frac{t^n \lambda^{2n}}{n!} \|R(\lambda)^n\| \leq M e^{-\lambda t} \sum_n \frac{t^n \lambda^{2n}}{n!(\lambda - a)^n}$$
$$= M e^{t \frac{a\lambda}{\lambda - a}} \leq M e^{2at}.$$

Also for $\lambda \to \infty$,

$$\limsup \|T_\lambda(t)\| \leq M e^{at}. \tag{1}$$

Claim. $T_\lambda(t)$ converge in the strong operator topology (as $\lambda \to \infty$), uniformly for t in bounded intervals.

For $x \in D(A)$ and $\lambda, \mu > 2a$,

$$\|T_\mu(t)x - T_\lambda(t)x\| = \left\| \int_0^t \frac{d}{ds}[T_\lambda(t-s)T_\mu(s)x]\, ds \right\|$$
$$= \left\| \int_0^t T_\lambda(t-s)T_\mu(s)(A_\mu - A_\lambda)x\, ds \right\|$$
$$\leq M^2 e^{4at} t \|A_\mu x - A_\lambda x\| \to 0$$

when $\lambda, \mu \to \infty$, by Lemma 1.16, uniformly for t in bounded intervals.

Since $\|T_\lambda(\cdot)\|$ is uniformly bounded in bounded intervals (by (1)), it follows from the density of $D(A)$ that $\{T_\lambda(t)x\}$ is Cauchy (as $\lambda \to \infty$) for all $x \in X$, uniformly for t in bounded intervals.

Define therefore

$$T(t)x = \lim_{\lambda \to \infty} T_\lambda(t)x \tag{2}$$

for all $x \in X$ (limit in X-norm).

By (1), $\|T(t)\| \leq M e^{at}$ for all $t \geq 0$. The semigroup property of $T(\cdot)$ follows from that of $T_\lambda(\cdot)$. The uniform convergence on bounded intervals implies the (strong) continuity of $T(\cdot)x$ on $[0, \infty)$, for each $x \in X$. Let A' denote the generator of $T(\cdot)$. We have

$$T_\lambda x - x = \int_0^t T_\lambda(s) A_\lambda x\, ds.$$

For $x \in D(A)$, Lemma 1.16 implies (by letting $\lambda \to \infty$)

$$T(t)x - x = \int_0^t T(s) A x\, ds.$$

Dividing by $t > 0$ and letting $t \to 0+$, we conclude that $x \in D(A')$ and $A'x = Ax$. Thus, for $\lambda > a$, $\lambda I - A$ and $\lambda I - A'$ are both one-to-one and onto X, and coincide on $D(\lambda I - A) = D(A)$. Therefore $D(A) = D(A')$, and the proof is complete. □

For *contraction semigroups* (i.e., $\|T(\cdot)\| \leq 1$), the Hille–Yosida characterization is especially simple (case $M = 1, a = 0$).

Corollary 1.18. *An operator A is the generator of a C_o-contraction semigroup iff it is densely defined, and $\lambda R(\lambda; A)$ (exist and) are contractions for all $\lambda > 0$.*

We call the bounded operators A_λ the *Hille–Yosida approximations* of A. From Lemma 1.16 and the proof of the Hille–Yosida theorem, $A_\lambda x \to Ax$ for all $x \in D(A)$ and $e^{tA_\lambda} \to T(t)$ strongly, uniformly on bounded t-intervals (as $\lambda \to \infty$).

A.11 The Hille–Yosida Space

The inequalities (2) following the proof of Lemma 1.16 can be used to construct, for an *arbitrary* (unbounded) operator A with $(a, \infty) \subset \rho(A)$, a maximal Banach subspace Z of X such that A_Z, the *part of A in Z*, generates a C_o-semigroup in Z.

Definition 1.19.

1. *A Banach subspace Y of X is a linear manifold $Y \subset X$ which is a Banach space for a norm $\| \cdot \|_Y \geq \| \cdot \|$.*
2. *If A is any operator on X with domain $D(A)$, and W is a linear manifold in X, the part of A in W, denoted A_W, is the restriction of A to its maximal domain as an operator in W:*

$$D(A_W) = \{x \in D(A); x, Ax \in W\}.$$

Definition 1.20. *Let A be an arbitrary operator with $(a, \infty) \subset \rho(A)$ for some $a \geq 0$. Denote*

$$\|x\|_Y = \sup \left\| \prod_k (\lambda_k - a) R(\lambda_k; A) x \right\|,$$

where the supremum is taken over all finite subsets $\{\lambda_1, \ldots, \lambda_m\}$ of (a, ∞) (the product over the empty set is defined as x). Set

$$Y = \{x \in X; \|x\|_Y < \infty\}.$$

Lemma 1.21. *The space $Y = (Y, \| \cdot \|_Y)$ is a Banach subspace of X, invariant under any bounded operator U which commutes with A, and $\|U\|_{B(Y)} \leq \|U\|_{B(X)}$.*

Proof. Clearly, Y is a linear manifold in X, and its norm majorizes $\| \cdot \|$. In particular, if $\{x_n\}$ is Cauchy in Y, it is also Cauchy in X; let x be its X-limit, and let $K = \sup_n \|x_n\|_Y$ ($< \infty$, because the sequence $\{x_n\}$ is Cauchy in Y). For any finite set $\{\lambda_k\}_{1 \leq k \leq m} \subset (a, \infty)$,

$$\left\|\prod_k (\lambda_k - a)R(\lambda_k;A)x\right\| = \lim_n \left\|\prod_k (\lambda_k - a)R(\lambda_k;A)x_n\right\|$$
$$\leq \limsup_n \|x_n\|_Y \leq K,$$

so that $\|x\|_Y \leq K < \infty$, i.e., $x \in Y$.

Given $\epsilon > 0$, there exists n_o such that $\|x_n - x_p\|_Y < \epsilon$ whenever $n, p > n_o$. Therefore for any finite set $\{\lambda_k\}$ as before,

$$\left\|\prod_k (\lambda_k - a)R(\lambda_k;A)(x_n - x_p)\right\| \leq \|x_n - x_p\|_Y < \epsilon$$

if $n, p > n_o$. Letting $p \to \infty$, and then taking the supremum over all finite subsets $\{\lambda_k\}$, we obtain $\|x_n - x\|_Y \leq \epsilon$ for all $n > n_o$. Thus Y is a Banach subspace of X.

If $U \in B(X)$ commutes with A, it commutes also with $R(\lambda;A)$ for each $\lambda > a$. Therefore, for $x \in Y$,

$$\|Ux\|_Y = \sup_{\lambda_k > a} \left\|U\prod_k (\lambda_k - a)R(\lambda_k;A)x\right\| \leq \|U\|_{B(X)}\|x\|_Y < \infty,$$

and so Y is U-invariant and $\|U\|_{B(Y)} \leq \|U\|_{B(X)}$. □

Definition 1.22. *The Hille–Yosida space Z for A is the closure of $D(A_Y)$ in Y.*

The terminology is motivated by the following:

Theorem 1.23. *Let A be an unbounded operator with $(a,\infty) \subset \rho(A)$ for some $a \geq 0$. Let Z be the Hille–Yosida space for A. Then A_Z, the part of A in Z, generates a C_o-semigroup $T(\cdot)$ in Z that satisfies $\|T(t)\|_{B(Z)} \leq e^{at}$.*

Moreover, Z is "maximal" in the following sense: if $W = (W, \|\cdot\|_W)$ is a Banach subspace of X such that A_W generates a C_o-semigroup in W with the above exponential growth, then W is a Banach subspace of Z.

Proof. Since $R(\lambda;A)$ commutes with A for each $\lambda > a$, the linear manifold Y is $R(\lambda;A)$-invariant and $\|R(\lambda;A)|_Y\|_{B(Y)} \leq \|R(\lambda;A)\|_{B(X)} < \infty$ by Lemma 1.21. The identities

$$(\lambda I - A)R(\lambda;A)y = y \quad (y \in Y)$$
$$R(\lambda;A)(\lambda I - A)y = y \quad (y \in D(A_Y))$$

show then that $R(\lambda;A)|_Y = R(\lambda;A_Y)$ for all $\lambda > a$.

If $y \in D(A_Y)$, then $y, Ay \in Y$, so that $R(\lambda;A)y (\in D(A)) \in Y$ and $AR(\lambda;A)y = \lambda R(\lambda;A)y - y \in Y$, that is, $R(\lambda;A)D(A_Y) \subset D(A_Y)$. Since $R(\lambda;A)|_Y \in B(Y)$, it follows that Z is $R(\lambda;A)$-invariant, and

$$\|R(\lambda; A)|_Z\|_{B(Z)} \leq \|R(\lambda; A)|_Y\|_{B(Y)} < \infty. \tag{1}$$

The above identities show then that

$$R(\lambda; A_Z) = R(\lambda; A)|_Z \quad (\lambda \in (a, \infty)). \tag{2}$$

In particular, A_Z is closed.

For all $y \in Y$ and all finite sets $\{\lambda_k\} \subset (a, \infty)$,

$$\begin{aligned}
&\left\|\prod_k (\lambda_k - a) R(\lambda_k; A_Y) y\right\|_Y \\
&\quad = \left\|\prod_k (\lambda_k - a) R(\lambda_k; A) y\right\|_Y \\
&\quad = \sup_{\mu_j > a} \left\|\prod_j (\mu_j - a) R(\mu_j; A) \prod_k (\lambda_k - a) R(\lambda_k; A) y\right\| \\
&\quad \leq \sup_{\nu_r > a} \left\|\prod_r (\nu_r - a) R(\nu_r; A) y\right\| = \|y\|_Y.
\end{aligned}$$

Therefore

$$\left\|\prod_k (\lambda_k - a) R(\lambda_k; A_Y)\right\|_{B(Y)} \leq 1 \tag{3}$$

for any finite set $\{\lambda_k\} \subset (a, \infty)$, and the same is true with Y replaced by Z. In particular, taking singleton subsets of (a, ∞), we have

$$\|R(\lambda; A_Y)\|_{B(Y)} \leq \frac{1}{\lambda - a} \quad (\lambda > a). \tag{4}$$

Therefore, for all $z \in D(A_Y)$,

$$\begin{aligned}
\|\lambda R(\lambda; A) z - z\|_Y = \|R(\lambda; A) A z\|_Y &\leq \|R(\lambda; A_Y)\|_{B(Y)} \, \|Az\|_Y \\
&\leq \frac{\|Az\|_Y}{\lambda - a} \to 0
\end{aligned}$$

as $\lambda \to \infty$, since $Az \in Y$. Thus $\lambda R(\lambda; A) z \to z$ in Y for all $z \in D(A_Y)$.

For $z \in Z$ arbitrary, if $\epsilon > 0$ is given, there exists $z_o \in D(A_Y)$ such that$\|z - z_o\|_Y < \epsilon$, since $D(A_Y)$ is dense in Z by Definition 1.22. Then

$$\begin{aligned}
\|\lambda R(\lambda; A_Z) z - z\|_Y &\leq \|(\lambda R(\lambda; A_Z) - I)(z - z_o)\|_Y + \|\lambda R(\lambda; A) z_o - z_o\|_Y \\
&\leq \left(\frac{\lambda}{\lambda - a} + 1\right) \epsilon + \|\lambda R(\lambda; A) z_o - z_o\|_Y \to 2\epsilon
\end{aligned}$$

as $\lambda \to \infty$. Hence as $\lambda \to \infty$,

$$\lambda R(\lambda; A_Z)z(\in D(A_Z)) \to z$$

in the $\|\cdot\|_Y$-norm, and so $D(A_Z)$ is dense in Z.

In conclusion, A_Z satisfies in Z the conditions of the Hille–Yosida theorem (with $M = 1$). Therefore A_Z generates in Z a C_o-semigroup $T(\cdot)$ satisfying $\|T(t)\|_Z \le e^{at}$ for all $t \ge 0$.

On the other hand, if W is as in the statement of the theorem, then for any $w \in W$, the estimates (2) following Lemma 1.16 (with $M = 1$), applied in the space $B(W)$, imply that

$$\|w\|_Y \le \sup_{\lambda_k > a} \left\| \prod_k (\lambda_k - a) R(\lambda_k; A) \right\|_{B(W)} \|w\|_W \le \|w\|_W.$$

Therefore W is a Banach subspace of Y. In particular $D(A_W) \subset D(A_Y)$. Since A_W generates a C_o-semigroup in W,

$$\begin{aligned} W = W - \mathit{closure}\,(D(A_W)) \subset W - \mathit{closure}\,(D(A_Y)) \\ \subset Y - \mathit{closure}\,(D(A_Y)) := Z, \end{aligned}$$

and we conclude that W is a Banach subspace of Z. □

Note in particular the case $a = 0$: if $(0, \infty) \subset \rho(A)$, the Hille–Yosida space for A is a maximal Banach subspace such that the part of A in it generates a C_o-semigroup of contractions in it.

A.12 Dissipative Operators

The purpose of this section is to present a useful characterization of generators of C_o-semigroups of *contractions* that avoids resolvents, and relies instead on the concept of a *dissipative operator*. The latter is based on a geometric property of the operator's *numerical range*.

Definition 1.24. *Let A be an arbitrary (usually unbounded) operator on the Banach space X. Its numerical range is the set*

$$\nu(A) = \{x^* Ax; x \in D(A), x^* \in X^*, \|x\| = \|x^*\| = x^* x = 1\}.$$

Given $x \in D(A)$ with $\|x\| = 1$, define x^* on $\mathbb{C}x$ by $x^*(\lambda x) = \lambda$ for $\lambda \in \mathbb{C}$. Then $\|x^*\| = x^* x = 1$, and x^* extends to a unit vector in X^* by the Hahn–Banach theorem. This shows that $\nu(A)$ is not empty.

Definition 1.25. *The operator A is dissipative if $\Re\nu(A) \le 0$.*

Theorem 1.26. *If A generates a C_o-semigroup of contractions, then it is closed, densely defined, dissipative, and $\lambda I - A$ is surjective for all $\lambda > 0$.*

Conversely, if A is closed, densely defined, dissipative, and $\lambda I - A$ is surjective for all $\lambda > \lambda_o$ (for some $\lambda_o \geq 0$), then A generates a C_o-semigroup of contractions.

Proof. Necessity. Suppose A generates the C_o-semigroup of contractions $T(\cdot)$, and let $x \in X$ and $x^* \in X^*$ be unit vectors such that $x^*x = 1$. For $h > 0$, $|x^*T(h)x| \leq \|x^*\| \, \|T(h)\| \, \|x\| \leq 1$, and therefore

$$\Re x^*[h^{-1}(T(h)x - x)] = h^{-1}[\Re(x^*T(h)x) - 1] \leq 0.$$

For $x \in D(A)$, letting $h \to 0$, we get $\Re x^*Ax \leq 0$, so that A is dissipative. It is closed and densely defined by Theorem 1.2. By Corollary 1.18, $(0, \infty) \subset \rho(A)$, so that, in particular, $\lambda I - A$ is surjective for all $\lambda > 0$.

Sufficiency. For all unit vectors $x \in D(A), x^* \in X^*$ such that $x^*x = 1$, and for all $\lambda > 0$, we have

$$\begin{aligned} \|(\lambda I - A)x\|^2 &\geq |x^*(\lambda I - A)x|^2 = |\lambda - x^*Ax|^2 \\ &= \lambda^2 - 2\lambda\Re(x^*Ax) + |x^*Ax|^2 \geq \lambda^2 \qquad (1) \end{aligned}$$

because $\Re(x^*Ax) \leq 0$. Therefore $\lambda I - A$ is one-to-one (for all $\lambda > 0$) and onto X (by hypothesis) for all $\lambda > \lambda_o$. Thus $\lambda \in \rho(A)$, and $\|\lambda R(\lambda; A)\| \leq 1$ for all $\lambda > \lambda_o$. This proves that A generates a C_o-contraction semigroup, by Corollary 1.18 (it is clear from the proof of Theorem 1.17 that for the sufficiency part, the growth condition on the resolvents is needed for large λ only). □

Note that in (1), we used only *some* unit vector x^* with the needed properties. This allows the following weakening of the hypothesis in the sufficiency part of the theorem.

Theorem 1.27. *Let A be a closable densely defined operator such that $\lambda_o I - A$ has dense range for some $\lambda_o > 0$. Suppose that for each $x \in D(A)$, there exists a unit vector $x^* \in X^*$ such that $x^*x = \|x\|$ and $\Re(x^*Ax) \leq 0$. Then the closure $\overline{A}$ of A generates a C_o-semigroup of contractions.*

Proof. As in (1), $\|(\lambda I - A)x\| \geq \lambda\|x\|$ for all $\lambda > 0$ and $x \in D(A)$. Let $x \in D(\overline{A})$, and let then $x_n \in D(A)$ be such that $x_n \to x$ and $Ax_n \to \overline{A}x$. Letting $n \to \infty$ in the inequalities $\|(\lambda I - A)x_n\| \geq \lambda\|x_n\|$, we obtain

$$\|(\lambda I - \overline{A})x\| \geq \lambda\|x\| \quad (\lambda > 0; x \in D(\overline{A})). \qquad (2)$$

In particular, $\lambda I - \overline{A}$ is one-to-one for all $\lambda > 0$. We claim that $\lambda_o I - \overline{A}$ is onto X. Indeed, for any $y \in X$, there exist by hypothesis $x_n \in D(A)$ such that $(\lambda_o I - A)x_n \to y$. Then by (2),

$$\|x_n - x_m\| \leq \lambda_o^{-1}\|(\lambda_o I - A)(x_n - x_m)\| \to 0,$$

so $x_n \to x$, and necessarily $x \in D(\overline{A})$ and $(\lambda_o I - \overline{A})x = y$.

Thus $\lambda_o \in \rho(\overline{A})$ and $\|R(\lambda_o; \overline{A})\| \leq 1/\lambda_o$. By Theorem 1.11,

$$d(\lambda_o, \sigma(\overline{A})) \geq \frac{1}{\|R(\lambda_o; \overline{A})\|} \geq \lambda_o.$$

Therefore $(0, 2\lambda_o) \subset \rho(\overline{A})$. Inductively, one obtains that $(0, 2^n\lambda_o) \subset \rho(\overline{A})$ for all n, and so $(0, \infty) \subset \rho(\overline{A})$ and $\lambda R(\lambda; \overline{A})$ are contractions for all $\lambda > 0$, by (2). The result follows now from Corollary 1.18. □

The criterion of Theorem 1.26 is effective for certain types of "perturbations" of generators.

Definition 1.28. *Let A, B be (usually unbounded) operators. One says that B is A-bounded if $D(A) \subset D(B)$ and there exist $a, b \geq 0$ such that*

$$\|Bx\| \leq a\|Ax\| + b\|x\| \quad (x \in D(A)).$$

The infimum of all a as above is called the A-bound of B.

For example, any $B \in B(X)$ is A-bounded with A-bound equal to 0.

Lemma 1.29. *If A is closed and B is A-bounded with A-bound $a < 1$, then $A + B$ (with domain $D(A)$) is closed.*

Proof. Note first that the A-boundedness of B means that

$$B : [D(A)] \to X$$

is *continuous* (recall that $[D(A)]$ is normed by the graph-norm for A).

Let $x_n \in D(A), x_n \to x$, and $(A + B)x_n \to y$. Then

$$\begin{aligned}\|Ax_n - Ax_m\| &= \|(A+B)x_n - (A+B)x_m - B(x_n - x_m)\| \\ &\leq \|(A+B)x_n - (A+B)x_m\| + a\|Ax_n - Ax_m\| + b\|x_n - x_m\|.\end{aligned}$$

Hence

$$(1 - a)\|Ax_n - Ax_m\| \leq \|(A+B)x_n - (A+B)x_m\| + b\|x_n - x_m\| \to 0$$

as $n, m \to \infty$. Since $a < 1$, $\{Ax_n\}$ is Cauchy, and since A is closed, it follows that $x \in D(A)(= D(A + B))$ and $Ax_n \to Ax$. Since $B : [D(A)] \to X$ is continuous and $x_n \to x$ in $[D(A)]$, it follows that $Bx_n \to Bx$, and therefore $(A + B)x_n \to (A + B)x$. □

We now have the following perturbation theorem.

Theorem 1.30. *Let A generate a C_o-semigroup of contractions, and let B be dissipative and A-bounded with A-bound $a < 1$. Then $A + B$ generates a C_o-semigroup of contractions.*

Proof. Since A generates a C_o-contraction semigroup, it is dissipative (1.26). Also B is dissipative with $D(A) \subset D(B)$. Therefore $A + B$ is dissipative, because for all $x \in D(A+B) = D(A)$ and $x^* \in X^*$ with $\|x\| = \|x^*\| = x^*x = 1$, $\Re[x^*(A+B)x] = \Re[x^*Ax] + \Re[x^*Bx] \leq 0$. By Lemma 1.29, $A+B$ is closed, and it is densely defined ($D(A)$ is dense by Theorem 1.2). By Theorem 1.26, it remains to show that $\lambda I - (A + B)$ is surjective for all $\lambda > \lambda_o$ (for some $\lambda_o \geq 0$). We have for all $\lambda > 0$

$$\operatorname{ran}(\lambda I - A - B) = [(\lambda I - A) - B]R(\lambda; A)X = [I - BR(\lambda; A)]X. \quad (3)$$

However, for all $x \in X$,

$$\begin{aligned} \|BR(\lambda; A)x\| &\leq a\|AR(\lambda; A)x\| + b\|R(\lambda; A)x\| \\ &\leq a\|\lambda R(\lambda; A)\| \, \|x\| + a\|x\| + \frac{b}{\lambda}\|\lambda R(\lambda; A)\| \, \|x\| \\ &\leq \left(2a + \frac{b}{\lambda}\right) \|x\| \end{aligned}$$

since $\lambda R(\lambda; A)$ are contractions, by Corollary 1.18.

In case $a < 1/2$, $2a + \frac{b}{\lambda} < 1$ for $\lambda > \lambda_o$, and therefore $\|BR(\lambda; A)\| < 1$, hence $I - BR(\lambda; A)$ is invertible in $B(X)$, and so $\lambda I - (A + B)$ is surjective for $\lambda > \lambda_o$, by (3).

Consider now the general case $a < 1$. Let $t_1 = \frac{1-a}{2}$. For any $s \in [0, 1]$ and $x \in D(A)$,

$$\begin{aligned} (1 - as)\|Bx\| &= \|Bx\| - as\|Bx\| \leq (a\|Ax\| + b\|x\|) - as\|Bx\| \\ &= a(\|Ax\| - s\|Bx\|) + b\|x\| \leq a\|(A + sB)x\| + b\|x\|. \end{aligned}$$

Therefore, for $t \in [0, t_1]$,

$$\|tBx\| \leq \frac{1 - as}{2}\|Bx\| \leq \frac{a}{2}\|(A + sB)x\| + \frac{b}{2}\|x\|,$$

so that tB has an $(A + sB)$-bound $< \frac{1}{2}$. By the preceding case, if $A + sB$ generates a C_o-contraction semigroup for some $s \in [0, 1]$, so does $A + sB + tB$ for all $t \in [0, t_1]$. Starting with $s = 0$ (for which $A + sB = A$ generates a C_o-semigroup of contractions by hypothesis), we get that $A + tB$ generates a C_o-semigroup of contractions for all $t \in [0, t_1]$, hence $A + t_1B + tB$ is such a generator for all such t, etc. Let n be the first integer such that $nt_1 \geq 1$. A last application of the above argument with $s = (n - 1)t_1 < 1$ gives that $A + tB$ is the generator of a C_o-contraction semigroup for all $t \in [0, nt_1]$, so that, in particular, $A + B$ is such a generator. □

A.13 The Trotter–Kato Convergence Theorem

The key ingredient in the proof of the Hille–Yosida theorem was the approximation of the *unbounded* operator A by the so-called Hille–Yosida approximations A_λ, which were *bounded* operators converging to A in some sense as $\lambda \to \infty$. The semigroup $T(\cdot)$ generated by A was then obtained as the strong limit of the semigroups $T_\lambda(t) := e^{tA_\lambda}$ (as $\lambda \to \infty$). Cf. Equation (2) in Section A.10.

In many practical situations, the operator A may be approximated by "simpler" operators. For example, a partial differential operator may be approximated (at least locally) by its term of highest order, which may luckily have *constant* coefficients.

This motivates the study in this section of the relation between generators convergence (or "approximation") and strong convergence of the corresponding semigroups. In the language of the associated abstract Cauchy problems, we are interested here with the relation between coefficients approximation and solutions approximation.

Let $T_s(\cdot)$ $(0 \le s < c)$ be C_o-semigroups with generators A_s. (Write $T(\cdot) = T_0(\cdot)$ and $A = A_0$.)

Basic Hypothesis: there exist constants $M > 0$ and $a \ge 0$ such that

$$\|T_s(t)\| \le Me^{at} \quad (t \ge 0, s \in [0, c)). \tag{1}$$

This implies (cf. (3) following 1.15)

$$\|R(\lambda; A_s)\| \le \frac{M}{\lambda - a} \tag{2}$$

for all $\lambda > a$ and $s \in [0, c)$.

We define the following convergence properties:

Definition 1.31.

1. *Generators graph convergence on a core D_o for A: for each $x \in D_o$, there exists $x_s \in D(A_s)$ such that $[x_s, A_s x_s] \to [x, Ax]$ in X^2 when $s \to 0$.*
2. *Resolvents strong convergence: for each $\lambda > a$, $R(\lambda; A_s) \to R(\lambda; A)$ in the strong operator topology (when $s \to 0$).*
3. *Semigroups strong uniform convergence on compacta: for each $x \in X$, $T_s(t)x \to T(t)x$ in X, uniformly on compact t-intervals (when $s \to 0$).*

Theorem 1.32 (The Trotter–Kato Theorem). *The convergence properties in Definition 1.31 are equivalent.*

Proof. In the following, the numbers 1, 2, 3 refer to the three types of convergence formulated in Definition 1.31 and limits are for $s \to 0+$.

Denoting $y_s = (\lambda I - A_s)x_s$ and $y = (\lambda I - A)x$ for all $x \in D_o$, we see that Property 1 is equivalent to

1′. $[y_s, R(\lambda; A_s)y_s] \to [y, R(\lambda; A)y]$ for all $y \in (\lambda I - A)D_o$ (for suitable y_s, and for all $\lambda > a$).

By (2), Property 1′ is equivalent to

$1''$. $[y_s, R(\lambda; A_s)y] \to [y, R(\lambda; A)y]$ for all $y \in (\lambda I - A)D_o$ (for suitable y_s, and for all $\lambda > a$).

1 implies 2. Since we are assuming $1''$, we have in particular

$$R(\lambda; A_s)y \to R(\lambda; A)y \tag{3}$$

for all $y \in (\lambda I - A)D_o$. Since D_o is a core for A, for each $x \in D(A)$, there exist $x_n \in D_o, n = 1, 2, \ldots$, such that $[x_n, Ax_n] \to [x, Ax]$, hence $[x_n, (\lambda I - A)x_n] \to [x, (\lambda I - A)x]$. In particular, $X = (\lambda I - A)D(A) = \overline{(\lambda I - A)D_o}$.

Thus (3) is valid for y in a dense subspace of X, hence for all $y \in X$, by (2). This is Property 2.

2 implies 1. Given $x \in D_o$ and $\lambda > a$, choose $x_s = R(\lambda; A_s)(\lambda I - A)x$. Correspondingly, we have $y = (\lambda I - A)x$ and $y_s = (\lambda I - A)x = y$, so that by Property 2

$$[y_s, R(\lambda; A_s)y] = [y, R(\lambda; A_s)y] \to [y, R(\lambda; A)y].$$

Thus $1''$ (and so 1) is satisfied.

2 implies 3. We need the following.

Lemma. *Let A, B generate the C_o-semigroups $T(\cdot)$ and $V(\cdot)$, respectively, both $O(e^{at})$ for some $a \geq 0$. Then for $\Re\lambda > a, t \geq 0$ and $x \in X$,*

$$R(\lambda; B)[V(t) - T(t)]R(\lambda; A)x = \int_0^t V(t-s)[R(\lambda; B) - R(\lambda; A)]T(s)x\, ds.$$

Proof of Lemma. For λ, t as above and $0 \leq s \leq t$,

$$\begin{aligned}
&\frac{d}{ds} V(t-s)R(\lambda; B)T(s)R(\lambda; A)x \\
&\quad = V(t-s)(-B)R(\lambda; B)T(s)R(\lambda; A)x \\
&\qquad + V(t-s)R(\lambda; B)T(s)AR(\lambda; A)x \\
&\quad = V(t-s)[T(s)R(\lambda; A)x - \lambda R(\lambda; B)T(s)R(\lambda; A)x] \\
&\qquad + V(t-s)R(\lambda; B)T(s)[\lambda R(\lambda; A)x - x] \\
&\quad = V(t-s)[R(\lambda; A) - R(\lambda; B)]T(s)x.
\end{aligned}$$

Integrating with respect to s from 0 to t, we obtain the formula in the lemma.

Now, for all $y \in X$, we write

$$\begin{aligned}
&[T_s(t) - T(t)]R(\lambda; A)y \\
&\quad = R(\lambda; A_s)[T_s(t) - T(t)]y \\
&\qquad + T_s(t)[R(\lambda; A) - R(\lambda; A_s)]y \\
&\qquad + [R(\lambda; A_s) - R(\lambda; A)]T(t)y = I + II + III \quad (s, t \geq 0, \lambda > a).
\end{aligned}$$

We estimate I for $y = R(\lambda; A)x$, using the lemma. Thus for $0 \le t \le \tau$,

$$\|I\| \le \int_0^\tau M e^{a(\tau-u)} \|[R(\lambda; A_s) - R(\lambda; A)]T(u)x\|\, du.$$

The right-hand side converges to zero when $s \to 0$, by 2; therefore $I \to 0$ uniformly on every compact t-interval. Since $R(\lambda; A)X = D(A)$ is dense in X, and the operators in I are uniformly bounded with respect to s (and with respect to t in compacta (cf. (1) and (2)), it follows that I converges to zero for all $y \in X$, uniformly on compact t-intervals (when $s \to 0$).

By (1) and 2, for $0 \le t \le \tau$,

$$\|II\| \le M e^{a\tau} \|R(\lambda; A)y - R(\lambda; A_s)y\| \to 0$$

as $s \to 0$, so that $II \to 0$ uniformly on compact t-intervals.

For $y \in D(A)$, we may write

$$T(t)y = y + \int_0^t T(r)Ay\, dr,$$

and therefore, for $0 \le t \le \tau$,

$$\|III\| \le \|[R(\lambda; A_s) - R(\lambda; A)]y\| + \int_0^\tau \|[R(\lambda; A_s) - R(\lambda; A)]T(r)Ay\|\, dr.$$

By 2, the first term on the right converges to 0 as $s \to 0$. The integrand on the right converges pointwise to 0, and is bounded by $\frac{2M^2}{\lambda - a} e^{a\tau} \|Ay\|$ in the interval $[0, \tau]$. By Lebesgue's Dominated Convergence Theorem, the integral converges to 0, and therefore $III \to 0$ uniformly on compact t-intervals. Since $D(A)$ is dense in X and the operators appearing in III are uniformly bounded (with respect to $s \in [0, c)$ and t in compacta; cf. (1) and (2)), the above conclusion is valid for all $y \in X$.

We thus obtained that $T_s(t)x \to T(t)x$ (when $s \to 0$), uniformly on compact t-intervals, for all $x \in R(\lambda; A)X = D(A)$ ($\lambda > a$ fixed); since $D(A)$ is dense in X and $\|T_s(t) - T(t)\| \le 2Me^{at}$ are uniformly bounded for all $s \ge 0$ and for all t in compacta, Property 3 follows.

3 implies 2. For $\lambda > a$ and $x \in X$, $R(\lambda; A_s)x = \int_0^\infty e^{-\lambda t} T_s(t)x\, dt$. When $s \to 0$, the integrand converges pointwise to its value at $s = 0$ (by 3) and is norm-dominated by $Me^{-(\lambda - a)t}\|x\| \in L^1(0, \infty)$. By Lebesgue's Dominated Convergence Theorem, the integral converges to its value at $s = 0$, which is precisely $R(\lambda; A)x$. □

Corollary 1.33 (Same "basic hypothesis" (1)). *Suppose that for each x in a core D_o for A, there exists $s_o \in (0, c)$ such that $x \in D(A_s)$ for all $s \in (0, s_o)$ and $A_s x \to Ax$ when $s \to 0+$. Then $T_s(t)x \to T(t)x$ for all $x \in X$, uniformly on compact t-intervals (when $s \to 0+$).*

Proof. Property 1 is satisfied with $x_s = x$ for $s \in (0, s_o)$. Therefore Property 3 holds, by Theorem 1.32. □

Corollary 1.34. *Let $T(\cdot)$ be a C_o-semigroup, and denote $A_s = s^{-1}[T(s) - I]$. Then*

$$T(t) = \lim_{s \to 0+} e^{tA_s}$$

strongly and uniformly on compact t-intervals.

Proof. Let $r > \omega$, and choose $a = re^r$. Let $T_s(t) = e^{tA_s}$ for $t \geq 0, s \in (0,1)$, and $T_0(\cdot) = T(\cdot)$. For $M = M_r$, we have $\|T(t)\| \leq Me^{rt} \leq Me^{at}$ and

$$\|T_s(t)\| = e^{-t/s}\|e^{(t/s)T(s)}\| = e^{-t/s}\left\|\sum_{n \geq 0}(t/s)^n T(ns)/n!\right\|$$

$$\leq Me^{-t/s}\sum(t/s)^n e^{nsr}/n! = M\exp[ts^{-1}(e^{sr} - 1)] \leq Me^{at}$$

for all $t \geq 0$ and $s \in (0,1)$.

Thus the "basic hypothesis" (1) is satisfied (with $c = 1$), and since $A_s x \to Ax$ for all $x \in D(A)$ (by definition of A), our corollary follows from Corollary 1.33. □

A.14 Exponential Formulas

If we write the calculus formula

$$e^{ta} = \lim_n \left(1 - \frac{at}{n}\right)^{-n} \qquad (a \in \mathbb{C}; t \in \mathbb{R})$$

in the form

$$e^{ta} = \lim_n \left[\frac{n}{t}R\left(\frac{n}{t}; a\right)\right]^n \qquad (t > 0),$$

the right-hand side makes sense also when the scalar a is replaced by the generator A of a C_o-semigroup $T(\cdot)$ (which should play the role of the "exponential" e^{tA}). We shall prove in this section that the expected "exponential formula" is in fact valid in the strong operator topology. Another trivial exponential formula, namely,

$$e^{t(a+b)} = e^{ta}e^{tb} \quad (a, b \in \mathbb{C}),$$

is false in general when the scalars a, b are replaced by generators A, B of semigroups, mainly because A, B need not commute in general. However, if we rewrite the above formula in the obvious "limit" form

$$e^{t(a+b)} = \lim_n \left[e^{(t/n)a}e^{(t/n)b}\right]^n,$$

it turns out to be correct (in the strong operator topology) for contraction C_o-semigroups $S(\cdot)$, $T(\cdot)$, $U(\cdot)$ (replacing the three exponentials above) with respective generators A, B, C such that $C = A + B$ on some core for C. This so-called "Trotter Product Formula" (as well as the exponential formula mentioned before) will be obtained as corollaries of a general limit theorem of Kato (Theorem 1.35), which is itself an application of Corollary 1.33 in the preceding section. We now start with Kato's general limit theorem.

Theorem 1.35. *Let A generate a C_o-contraction semigroup $T(\cdot)$, and let F be any contraction-valued function on $[0, \infty)$ such that $F(0) = I$ and the right derivative at zero of $F(\cdot)x$ coincides with Ax for all x in a core D_o for A. Then $T(\cdot)$ is the strong limit of $F(t/n)^n$ (as $n \to \infty$), uniformly on compact t-intervals.*

Proof. We need the following

Lemma. *Let C be a contraction on X. Then $e^{t(C-I)}$ is a uniformly continuous contraction semigroup, and*

$$\|e^{n(C-I)}x - C^n x\| \leq n^{1/2}\|(C-I)x\|$$

for all $x \in X$ and $n = 1, 2, \ldots$.

Proof of Lemma. The operators $e^{t(C-I)}$ are contractions because

$$\|e^{t(C-I)}\| = e^{-t}\|e^{tC}\| \leq e^{-t}e^{t\|C\|} \leq 1.$$

Next we have

$$\begin{aligned}
&\|e^{n(C-I)}x - C^n x\| \\
&\quad \leq e^{-n}\left\|\sum_{k\geq 0}(n^k/k!)[C^k x - C^n x]\right\| \\
&\quad = e^{-n}\left\|\sum_{0\leq k\leq n}(n^k/k!)C^k(x - C^{n-k}x) + C^n\sum_{k>n}(n^k/k!)(C^{k-n}x - x)\right\| \\
&\quad \leq e^{-n}\sum_{k\geq 0}(n^k/k!)\|C^{|k-n|}x - x\|.
\end{aligned}$$

Since C is a contraction,

$$\|C^m x - x\| = \|(C^{m-1} + \cdots + I)(C-I)x\| \leq m\|Cx - x\|,$$

and therefore the last expression is

$$\leq \left[\sum_{k\geq 0} e^{-n}(n^k/k!)|k-n|\right]\|Cx - x\|.$$

The term in square brackets is the expectation of $|K-n|$, where K is a Poisson random variable with parameter n. By Schwarz' inequality, since K has expectation $E(K)$ and variance $\sigma^2(K)$ both equal to n, we have

$$E(|K-n|) \leq [E(K-n)^2]^{1/2} := \sigma(K) = n^{1/2}. \qquad \square$$

Back to the proof of the theorem, consider the bounded operators

$$A_n = (t/n)^{-1}[F(t/n) - I]$$

for t fixed. By hypothesis, $A_n x \to Ax$ for all $x \in D_o$. For all unit vectors $x \in X$ and $x^* \in X^*$ such that $x^* x = 1$, we have

$$\Re(x^* A_n x) = (n/t)[\Re(x^* F(t/n)x) - 1] \leq 0$$

because

$$|x^* F(\cdot)x| \leq \|x^*\| \, \|F(\cdot)\| \, \|x\| \leq 1.$$

Thus A_n is dissipative, and so, by 1.26, e^{sA_n} are contraction semigroups satisfying trivially the "basic hypothesis" (with $M=1$ and $a=0$). By Corollary 1.33,

$$e^{sA_n} \to T(s) \tag{1}$$

strongly and uniformly on compact s-intervals, when $n \to \infty$.

However, by the lemma with $C = F(t/n)$,

$$\|e^{tA_n}x - F(t/n)^n x\| \leq n^{1/2} \|[F(t/n) - I]x\| = \frac{t}{n^{1/2}} \|A_n x\| \to 0 \tag{2}$$

as $n \to \infty$, for all $x \in D_o$. Since D_o is dense in X (as a core) and $\|e^{tA_n} - F(t/n)^n\| \leq 2$ for all n because both operators are contractions, it follows that (2) is valid for all $x \in X$. By (1) and (2), the theorem follows. $\square$

As a first application of Theorem 1.35, we obtain the "exponential formula" mentioned at the beginning of this section.

Theorem 1.36 (Exponential formula). *Let $T(\cdot)$ be a C_o-semigroup, with generator A. Then for all $t > 0$,*

$$T(t) = \lim_n \left[\frac{n}{t} R\left(\frac{n}{t}; A\right)\right]^n$$

in the strong operator topology (as $n \to \infty$).

Proof. We consider first the case when $\|T(t)\| \leq e^{at}$ for all $t \geq 0$ (with some $a \geq 0$).

Let $S(t) := e^{-at}T(t)$. This is a C_o-semigroup of contractions, with generator $A - aI$.

We set

$$F(s) := (s^{-1} - a)R(s^{-1}; A) = (s^{-1} - a)R(s^{-1} - a; A - aI) \quad (s \in (0, 1/a)),$$

and $F(0) = I$.

By Corollary 1.18, $F(\cdot)$ is contraction-valued. By Lemma 1.16,

$$s^{-1}[F(s)x - x] = A_{1/s}x - as^{-1}R(s^{-1}; A)x \to (A - aI)x$$

for all $x \in D(A) = D(A - aI)$, as $s \to 0+$. We may then apply Theorem 1.35; thus, in the strong operator topology and uniformly in compact t-intervals,

$$S(t) = \lim_n F(t/n)^n,$$

and therefore

$$T(t) = e^{at}S(t) = \lim_n \left[\left(1 - \frac{at}{n}\right)^{-1} F(t/n)\right]^n$$

$$= \lim_n \left[\frac{n}{n - at}\frac{n - at}{t}R\left(\frac{n}{t}; A\right)\right]^n = \lim_n \left[\frac{n}{t}R\left(\frac{n}{t}; A\right)\right]^n.$$

General case. Fix $a > \omega$. We have $\|T(t)\| \leq Me^{at}$ for all $t \geq 0$ (with a suitable M depending only on a). Renorm X by $|x| := \sup_{t \geq 0} e^{-at}\|T(t)x\|$. Then $\|x\| \leq |x| \leq M\|x\|$, i.e., the norms are equivalent, and

$$|T(t)x| = \sup_{s \geq 0} e^{-as}\|T(s + t)x\| \leq e^{at} \sup_{u \geq 0} e^{-au}\|T(u)x\| = e^{at}|x|.$$

We may then apply the preceding case to the semigroup $T(\cdot)$ on the space $(X, |\cdot|)$, yielding the result with respect to the $|\cdot|$-norm, hence also with respect to the given norm (since the two norms are equivalent). □

Another application of Theorem 1.35 is the so-called "Trotter Product Formula" mentioned in the introductory observations of this section.

Theorem 1.37 (Trotter's Product Formula). *Let A, B, C generate contraction C_o-semigroups $S(\cdot), T(\cdot), U(\cdot)$, respectively, and suppose that $C = A + B$ on a core D_o for C. Then for all $t \geq 0$,*

$$U(t) = \lim_n [S(t/n)T(t/n)]^n$$

strongly.

Proof. Take $F(t) = S(t)T(t)$ in Theorem 1.35. For $x \in D_o$ and $t > 0$,

$$t^{-1}[F(t)x - x] = S(t)t^{-1}[T(t)x - x] + t^{-1}[S(t)x - x] \to Bx + Ax = Cx$$

as $t \to 0$, and the conclusion follows from Theorem 1.35. □

A.15 Perturbation of Generators

Another common way to look at the "simplification" of a Cauchy problem (corresponding to an operator A') by a Cauchy problem corresponding to a "simpler" operator A is to consider A' as a "perturbation" of A by some operator B, that is, $A' = A + B$, with B in a suitable class of operators. Since the C^1-solvability of the Cauchy problem for an operator A (with initial values in $D(A)$) is equivalent to A being the generator of a C_o-semigroup (cf. Theorem 1.2), the substance of the perturbation problem is the invariance of the generation property under perturbations by operators B in a sufficiently wide class. This problem is taken up in this subsection, namely:

Given the generator A of a C_o-semigroup $T(\cdot)$, formulate conditions on an operator B that are sufficient for the "perturbation" $A+B$ to be also the generator of a C_o-semigroup.

Hypothesis H_1. B is a closed operator such that $T(t)X \subset D(B)$ for all $t > 0$.

By the Closed Graph Theorem, $BT(t) \in B(X)$ for all $t > 0$. Hence, if $t, h > 0$, $BT(t+h)x - BT(t)x = [BT(t)][T(h)x - x] \to 0$ when $h \to 0$, showing that $BT(\cdot)$ is strongly right continuous on $(0, \infty)$. It follows that $\|BT(\cdot)\|$ is bounded on compact intervals, and we deduce that $BT(\cdot)$ is strongly continuous for $t > 0$, as in the proof of Theorem 1.1. Thus $\|BT(\cdot)\|$ is measurable on $t > 0$ (cf. proof of Theorem 1.2).

Also, for any $t > \epsilon > 0$,

$$\frac{\log \|BT(t)\|}{t} \le \frac{\log \|BT(\epsilon)\|}{t} + \left(1 - \frac{\epsilon}{t}\right) \frac{\log \|T(t-\epsilon)\|}{t-\epsilon} \to \omega,$$

so that

$$\limsup_{t\to\infty} \frac{\log \|BT(t)\|}{t} \le \omega.$$

(ω denotes as usual the type of the given semigroup $T(\cdot)$.)

Therefore, for any $a > \omega$, there exists a constant $M_a > 0$ such that $\|BT(t)\| \le M_a e^{at}$.

The nonnegative measurable function $\|BT(\cdot)\|$ has an integral over $[0, 1]$ (that could be infinite). We assume

Hypothesis H_2. $\int_0^1 \|BT(t)\|\, dt < \infty$.

(Note that any $B \in B(X)$ satisfies H_1 and H_2 trivially.)

Theorem 1.38 (The Hille–Phillips Perturbation Theorem). *Let A generate the C_o-semigroup $T(\cdot)$, and let B satisfy hypotheses H_1 and H_2. Then $A+B$ (with domain $D(A)$) generates a C_o-semigroup.*

Corollary. *If A generates a C_o-semigroup on the Banach space X, so does $A+B$ for all $B \in B(X)$.*

More details about the structure of the semigroup generated by the perturbation $A+B$ will be obtained in Lemma 3 and in (6) below.

Lemma 1 (Hypotheses H_1, H_2). *$D(A) \subset D(B)$, and for $\Re\lambda > \omega$,*

$$BR(\lambda; A)x = \int_0^\infty e^{-\lambda t} BT(t)x \, dt$$

for all $x \in X$, where the Laplace integral above converges absolutely.

Proof. Fix $x \in X$ and $\lambda \in \mathbb{C}$ such that $\Re\lambda > \omega$. The Riemann sums S_n for the integral $\int_a^b$ (with $0 < a < b < \infty$) approximating $\int_0^\infty e^{-\lambda t} T(t)x \, dt$ are in $D(B)$ by H_1, converge to $\int_a^b$, and by linearity of B and continuity of $BT(\cdot)x$ in $[a, b]$, BS_n converge to $\int_a^b e^{-\lambda t} BT(t)x \, dt$. Since B is closed, it follows that $\int_a^b \in D(B)$ and $B \int_a^b = \int_a^b e^{-\lambda t} BT(t)x \, dt$. Next

$$\int_0^\infty \|e^{-\lambda t} BT(t)\| \, dt \le e^{-\Re\lambda} \int_0^1 \|BT(t)\| \, dt + \int_1^\infty e^{-\Re\lambda t} \|BT(t)\| \, dt < \infty$$

by H_2 and the remarks following H_1. Consequently, the Laplace integral of $BT(\cdot)x$ converges absolutely in X to an element $L(\lambda)x \in X$. By Theorem 1.15, $\int_a^b e^{-\lambda t} T(t)x \, dt (\in D(B)) \to R(\lambda; A)x$ (when $a \to 0$ and $b \to \infty$), and we observed before that $B \int_a^b (\dots) \to L(\lambda)x$. Since B is closed, it follows that $R(\lambda; A)x \in D(B)$ and $BR(\lambda; A)x = L(\lambda)x$. Finally, every $y \in D(A)$ can be written in the form $y = R(\lambda; A)x$ with $\Re\lambda > \omega$ (take $x = (\lambda I - A)y$), and the inclusion $D(A) \subset D(B)$ follows. $\square$

Lemma 2 (Hypotheses H_1, H_2). *There exists $r > \omega$ such that*

$$q := \int_0^\infty e^{-rt} \|BT(t)\| \, dt < 1.$$

For $\Re\lambda > r$,

$$R(\lambda; A+B) = R(\lambda; A) \sum_{n \ge 0} [BR(\lambda; A)]^n, \tag{1}$$

where the series converges in $B(X)$.

In particular, $A+B$ is closed (and densely defined, since $D(A+B) = D(A)$, by Lemma 1).

Proof. Fix $c > \omega$. By H_1 (cf. discussion following the statement of Hypothesis H_1), for all $\lambda > c$, $e^{-\lambda t} \|BT(t)\| \le e^{-ct} \|BT(t)\| \in L^1(0, \infty)$, and $e^{-\lambda t} \|BT(t)\| \to 0$ as $\lambda \to \infty$. By Dominated Convergence, it follows that $\int_0^\infty e^{-\lambda t} \|BT(t)\| \, dt \to 0$ when $\lambda \to \infty$. We may then choose $r > c$ such that $q < 1$. Then, by Lemma 1, $\|BR(\lambda; A)\| \le q < 1$ for $\Re\lambda > r$, and therefore the right-hand side of (1) converges in $B(X)$ to an operator $K(\lambda)$ with range in $D(A) = D(A+B)$. We have for $\Re\lambda > r$

$$[\lambda I - (A+B)]K(\lambda) = (\lambda I - A)K(\lambda) - BK(\lambda)$$
$$= \sum_{n\geq 0}[BR(\lambda;A)]^n - \sum_{n\geq 0}[BR(\lambda;A)]^{n+1} = I.$$

On the other hand, for $x \in D(A)$,

$$K(\lambda)[\lambda I - (A+B)]x$$
$$= R(\lambda;A)\left\{I + \sum_{n\geq 1}[BR(\lambda;A)]^n\right\}[(\lambda I - A) - B]x$$
$$= x + R(\lambda;A)\left\{-Bx + \sum_{n\geq 0}[BR(\lambda;A)]^n[BR(\lambda;A)][(\lambda I - A) - B]x\right\}$$
$$= x + R(\lambda;A)\left\{-Bx + \sum_{n\geq 0}[BR(\lambda;A)]^n Bx - \sum_{n\geq 1}[BR(\lambda;A)]^n Bx\right\} = x.$$

□

The functions $f := \|T(\cdot)\|$ and $g := \|BT(\cdot)\|$ are both in L^1_{loc}, the class of locally integrable functions on $(0,\infty)$ (cf. remarks following H_1, together with H_2). The "Laplace" convolution

$$(u * v)(t) := \int_0^t u(t-s)v(s)\,ds \quad u, v \in L^1_{loc}$$

defines a function in L^1_{loc}, and therefore the repeated convolutions

$$g^{(n)} = g * \cdots * g \quad n \text{ times}$$

are in L^1_{loc}. We consider also the L^1_{loc}-functions $h^{(n)} = f * g^{(n)}$, and we set $h^{(0)} = f$.

The next lemma will justify the following inductive definition:

$$S_0(\cdot) = T(\cdot);$$
$$S_n(t)x = \int_0^t T(t-s)BS_{n-1}(s)x\,ds \quad (n = 1, 2, \ldots). \tag{2}$$

Lemma 3. *For all $n = 0, 1, 2, \ldots$, $S_n(\cdot)$ are well-defined bounded operators such that, for all $x \in X$,*

(a) *$S_n(\cdot)x : [0,\infty) \to D(B)$ is continuous, and for r, q as in Lemma 2 and all $t \geq 0$,*

$$\|S_n(t)\| \leq h^{(n)}(t) \leq Me^{rt}q^n;$$

(b) *$BS_n(\cdot)x : (0,\infty) \to X$ is continuous and $\|BS_n(t)\| \leq g^{(n+1)}(t)$.*

Proof. We prove the lemma by induction on n. The case $n = 0$ is trivial (see observations following H_1).

Assume that the lemma's claims are valid for some n. By (b) for n, $S_{n+1}(\cdot)$ is well-defined and

$$S_{n+1}(t)x = \lim \int_a^b T(t-s)BS_n(s)x\,ds \tag{3}$$

as $a \to 0+$ and $b \to t-$. The Riemann sums for each integral over $[a,b]$ are in $D(B)$ by H_1, and when B is applied to them, the new sums converge to $\int_a^b BT(t-s)BS_n(s)x\,ds$, because the latter's integrand is continuous on $[a,b]$ by (b) (for n). Since B is closed, it follows that each integral in (3) belongs to $D(B)$, and $B\int_a^b(\dots) = \int_a^b B(\dots)$. The same type of argument with $a \to 0+$ and $b \to t-$ (using again the closeness of B) shows that $S_{n+1}(t)x \in D(B)$ and

$$BS_{n+1}(t)x = \int_0^t BT(t-s)BS_n(s)x\,ds.$$

Since $T(\cdot)$, $BT(\cdot)$ and $BS_n(\cdot)$ are continuous on $(0,\infty)$ and majorized by L^1_{loc}-functions (using the induction hypothesis), the integral representations for $S_{n+1}(\cdot)x$ and $BS_{n+1}(\cdot)x$ imply their continuity on $(0,\infty)$ for each $x \in X$. The function $S_{n+1}(\cdot)$ is even norm-continuous at 0, since

$$\|S_{n+1}(t)\| \le f * g^{(n+1)} (= h^{(n+1)}) \le Me^{rt}\int_0^t g^{(n+1)}(s)\,ds \to 0$$

when $t \to 0+$, because the integrand is in L^1_{loc}. (Note that $S_n(0) = 0$ for all $n \ge 1$.)

The estimates in (a) and (b) for $n+1$ follow from the induction hypothesis. For example,

$$h^{(n+1)} = f * g^{(n+1)} = [f * g^{(n)}] * g = h^{(n)} * g,$$

so that

$$\begin{aligned} h^{(n+1)}(t) &\le Mq^n \int_0^t e^{r(t-s)} g(s)\,ds \\ &= Mq^n e^{rt}\int_0^t e^{-rs} g(s)\,ds \le Me^{rt}q^{n+1}. \end{aligned}$$ □

The exponential growth of $S_n(\cdot)$ (as in (a)) shows that the Laplace transform in the following lemma converges absolutely for $\Re\lambda > r$.

Lemma 4. *For $\Re\lambda > r$ and $x \in X$,*

$$R(\lambda;A)[BR(\lambda;A)]^n x = \int_0^\infty e^{-\lambda t} S_n(t)x\,dt \quad n = 0,1,2,\dots.$$

Proof. The case $n = 0$ is verified by Theorem 1.15. Assuming the lemma for n, we have by Theorem 1.15,

$$\begin{aligned} R(\lambda; A)[BR(\lambda; A)]^{n+1}x &= \int_0^\infty e^{-\lambda t} T(t)[BR(\lambda; A)]^{n+1}x \, dt \\ &= \int_0^\infty e^{-\lambda t} T(t)B \{R(\lambda; A)[BR(\lambda; A)]^n x\} \, dt \\ &= \int_0^\infty e^{-\lambda t} T(t)B \int_0^\infty e^{-\lambda s} S_n(s)x \, ds \, dt. \end{aligned}$$

(We used the induction hypothesis in the last equation.) Since B is closed, the argument we used before (cf. Lemmas 1 and 3) allows us to move B inside the inner integral, and then do the same with the bounded operator $T(t)$. We obtain the repeated integral

$$\begin{aligned} \int_0^\infty \int_0^\infty e^{-\lambda(t+s)} T(t)BS_n(s)x \, ds \, dt &= \int_0^\infty e^{-\lambda u} \int_0^u T(u-s)BS_n(s)x \, ds \, du \\ &= \int_0^\infty e^{-\lambda u} S_{n+1}(u)x \, du, \end{aligned}$$

where the interchange of integration order is justified by absolute convergence. □

We state now a simple characterization of generators by the property of their resolvent being the Laplace transform of a strongly continuous operator function with exponential growth.

Lemma 5. *An operator A on the Banach space X is the generator of a C_o-semigroup if and only if*

(a) *A is densely defined; and*
(b) *there exist constants $a \geq 0$ and $M > 0$, and a strongly continuous function $S(\cdot) : [0, \infty) \to B(X)$ such that $\|S(t)\| \leq M\, e^{at}$ for all $t \geq 0$ and*

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t} S(t)x \, dt \tag{4}$$

for all $\lambda > a$ and $x \in X$.

(When the conditions of the lemma are satisfied, $S(\cdot)$ is the semigroup generated by A.)

Proof. Necessity follows from Theorems 1.1, 1.2, and 1.15.

Sufficiency. The series expansion for the resolvent obtained in the proof of Theorem 1.11 shows that

$$(-1)^n R(\lambda; A)^{(n)} = n! R(\lambda; A)^{n+1}. \tag{5}$$

The exponential growth of $\|S(\cdot)\|$ allows us (using a Dominated Convergence argument) to differentiate the Laplace transform of $S(\cdot)x$ under the integral sign, yielding inductively to the formula

$$[R(\lambda; A)x]^{(n)} = \int_0^\infty (-t)^n e^{-\lambda t} S(t)x\, dt. \tag{6}$$

Therefore, for all real $\lambda > a$,

$$\begin{aligned}
\|R(\lambda; A)^n x\| &= \frac{1}{(n-1)!}\|[R(\lambda; A)x]^{(n-1)}\| \\
&\leq \frac{1}{(n-1)!}\int_0^\infty t^{n-1} e^{-\lambda t}\|S(t)\|\, dt\|x\| \\
&\leq \frac{M\|x\|}{(n-1)!}\int_0^\infty e^{-(\lambda-a)t} t^{n-1}\, dt = \frac{M}{(\lambda-a)^n}\|x\|,
\end{aligned}$$

and the Hille–Yosida theorem applies. If $T(\cdot)$ is the C_o-semigroup generated by A, then $R(\lambda; A)x$ is the Laplace transform of $T(\cdot)x$, and we conclude that $T(\cdot) = S(\cdot)$ by the Uniqueness Theorem for Laplace transforms. □

Proof of Theorem 1.38. For $0 \leq t \leq \tau$, we have by Lemma 3

$$\|S_n(t)\| \leq M e^{r\tau} q^n,$$

with $q < 1$. Therefore the series

$$S(t) := \sum_{n\geq 0} S_n(t) \tag{7}$$

converges in $B(X)$-norm, uniformly on every interval $[0, \tau]$. By Lemma 3, it follows that $S(\cdot)$ is strongly continuous on $[0, \infty)$, and $\|S(t)\| \leq \frac{M}{1-q}e^{rt}$.

Since $S_n(0) = 0$ for $n \geq 1$ (cf. Lemma 3), we have $S(0) = I$.

The exponential growth of $\|S(\cdot)\|$ shows that the Laplace transform of $S(\cdot)x$ converges absolutely for $\Re\lambda > r$, and a routine application of the Lebesgue Dominated Convergence Theorem shows that

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} S(t)x\, dt &= \sum_{n\geq 0}\int_0^\infty e^{-\lambda t} S_n(t)x\, dt \\
&= \sum_n R(\lambda; A)[BR(\lambda; A)]^n x = R(\lambda; A+B)x
\end{aligned}$$

by Lemmas 4 and 2, and we conclude now from Lemma 5 that $A+B$ generates the C_o-semigroup $S(\cdot)$ (given explicitly by (7) and Lemma 3). □

A.16 Groups of Operators

If the C_o-semigroup $T(\cdot)$ can be extended to $\mathbb{R} = (-\infty, \infty)$ with the semigroup identity extending to the *group identity*

$$T(s)T(t) = T(s+t) \quad (s, t \in \mathbb{R}),$$

it will be called a C_o*-group of operators*. (We refer also to the above extension as a group of operators.)

When this is the case, the semigroup $S(t) := T(-t)$, $t \geq 0$, is also of class C_o, since for $0 < t < \delta$,

$$S(t)x - x = T(-\delta)[T(\delta - t)x - T(\delta)x] \to 0$$

as $t \to 0+$, for all $x \in X$ (cf. Theorem 1.1).

The generator A' of $S(\cdot)$ is $-A$, because for $x \in D(A)$,

$$t^{-1}[S(t)x - x] = -T(-\delta)(-t)^{-1}[T(\delta - t)x - T(\delta)x] \to -Ax$$

as $t \to 0+$ (by Theorem 1.2), so that $-A \subset A'$, and therefore $A' = -A$ by symmetry.

It follows from Theorem 1.1 that $T(\cdot) : \mathbb{R} \to B(X)$ is a strongly continuous *representation* of the additive group $\mathbb{R}$ on X, with exponential growth as $|t| \to \infty$. Let ω, ω' be the types of $T(\cdot)$ and $S(\cdot)$, respectively. Since $\|S(t)\| = \|T(t)^{-1}\| \geq \|T(t)\|^{-1}$, we have

$$\omega' = \lim_{t\to\infty} t^{-1} \log \|S(t)\| \geq -\lim_{t\to\infty} t^{-1} \log \|T(t)\| = -\omega.$$

Note also that

$$\omega' = -\lim_{t\to-\infty} t^{-1} \log \|T(t)\|.$$

By Theorem 1.15, since $\sigma(-A) = -\sigma(A)$, the spectrum of A is necessarily contained in the closed strip

$$S : -\omega' \leq \Re\lambda \leq \omega. \tag{1}$$

Also, since $R(\lambda; -A) = -R(-\lambda; A)$, the necessary condition (for $a' > \omega'$ fixed and M a suitable constant depending on a')

$$\|R(\lambda; -A)^n\| \leq M/(\Re\lambda - a')^n$$

for $\Re\lambda > a'$ becomes $\|R(\lambda; A)^n\| \leq M/[(-a') - \Re\lambda]^n$ for $\Re\lambda < -a'$. Thus the growth condition on the resolvent outside the strip

$$S' : -a' \leq \Re\lambda \leq a$$

(where $a > \omega$ and $a' > \omega'$ are fixed) is

$$\|R(\lambda; A)^n\| \leq M/[d(\lambda, S')]^n, \tag{2}$$

where $M \geq 1$ is a constant depending on S'.

Relation (2) (assumed to hold for all *real* $\lambda \notin [-a', a]$ for some a, $a' \geq 0$), together with the density of $D(A)$, are *sufficient* for A to generate a C_o-group. Indeed, (2) implies that both A and $-A$ satisfy the conditions of the Hille–Yosida theorem. Let then $T(\cdot)$ and $S(\cdot)$ be the C_o-semigroups with the generators A and $-A$, respectively. Using Theorem 1.2, we have for all $x \in D(A)$:

$$\frac{d}{dt}[T(t)S(t)x] = T(t)AS(t)x + T(t)(-A)S(t)x = 0 \quad (t \geq 0).$$

Therefore $T(t)S(t) = T(0)S(0) = I$, and similarly $S(t)T(t) = I$, i.e., $S(t) = T(t)^{-1}$ for all $t \geq 0$. Thus A generates the C_o-group $T(\cdot)$ (its extension to $\mathbb{R}$ is trivially given by $T(-t) = T(t)^{-1}$ $(= S(t))$ for $t > 0$). Formally (with some change in notation)

Theorem 1.39. *The operator A generates a C_o-group of operators if and only if*

(a) *A is densely defined; and*
(b) *there exist constants a, $a' \geq 0$ and $M \geq 1$ such that, for all real $\lambda \notin [-a', a]$, $R(\lambda; A)$ exists and satisfies*

$$\|R(\lambda; A)^n\| \leq \frac{M}{\{d(\lambda; [-a', a])\}^n}$$

for all $n \in \mathbb{N}$.

A.17 Bounded Groups of Operators

In this subsection, we consider the special case when $T(\cdot)$ is a *bounded* C_o-group of operators on a *Hilbert space* X. We shall prove B. Sz.-Nagy's theorem to the effect that $T(\cdot)$ is *similar* to a C_o-group of *unitary* operators.

The "boundedness" assumption means that

$$\|T(t)\| \leq M \qquad (t \in \mathbb{R})$$

for some constant M (necessarily $M \geq \|T(0)\| = 1$). In this case (when X is an arbitrary Banach space), $\omega = \omega' = 0$, so that

$$\sigma(A) \subset i\mathbb{R}.$$

We consider now the case of a *Hilbert space* X (with inner product $(\cdot, \cdot)$).

Theorem 1.40 (B. Sz.-Nagy). *Let $T(\cdot)$ be a bounded C_o-group of operators acting on the Hilbert space X. Then there exists a nonsingular bounded (positive) operator Q such that $QT(\cdot)Q^{-1}$ is a group of unitary operators.*

Proof. Let $\mathbb{B}(\mathbb{R})$ denote the Banach algebra of all bounded complex functions on $\mathbb{R}$, with the supremum norm, and let LIM be the "generalized Banach limit" functional on it (cf. [DS I–III]). Define

$$(x, y)_T = \mathrm{LIM}(T(t)x, T(t)y) \quad (x, y \in X),$$

and let $\|x\|_T = (x, x)_T^{1/2}$. Then

$$M^{-1}\| \cdot \| \leq \| \cdot \|_T \leq M \| \cdot \|,$$

so that X is a Hilbert space under the equivalent inner product $(\cdot, \cdot)_T$, and there exists therefore a (strictly) positive operator P such that $(x, y)_T = (Px, y)$ for all $x, y \in X$. Let Q be the positive square root of P. For $s \in \mathbb{R}$ and $x, y \in X$ fixed, write $x = Qu$ and $y = Qv$. Then

$$\begin{aligned}
&(QT(s)Q^{-1}x, QT(s)Q^{-1}y) \\
&\quad = (PT(s)u, T(s)v) \\
&\quad = (T(s)u, T(s)v)_T = \mathrm{LIM}_t(T(t)T(s)u, T(t)T(s)v) \\
&\quad = \mathrm{LIM}_t(T(t+s)u, T(t+s)v) = \mathrm{LIM}(T(t)u, T(t)v) \\
&\quad = (u, v)_T = (Pu, v) = (Qu, Qv) = (x, y).
\end{aligned}$$

□

A.18 Stone's Theorem

In view of Theorem 1.40, the structure of bounded C_o-groups in Hilbert space is the "same" as that of C_o-groups of unitary operators ("up to similarity"). For the latter, we shall now prove the famous Stone representation theorem as an exponential e^{itH}, where H is a (generally unbounded) selfadjoint operator, and the exponential is defined by means of the operational calculus for such operators.

For the reader's convenience, we make a short digression about (unbounded) symmetric and selfadjoint operators on a Hilbert space X.

If A is a densely defined operator on X, its (Hilbert) *adjoint* A^* is defined as follows. The domain $D(A^*)$ of A^* is the set of all $y \in X$ for which there exists a (necessarily unique) $z \in X$ such that $(Ax, y) = (x, z)$ for all $x \in D(A)$; we then set $A^*y = z$ for all $y \in D(A^*)$. Thus

$$(Ax, y) = (x, A^*y) \quad (x \in D(A), y \in D(A^*)).$$

The *Hilbert adjoint* A^* of A is closed [indeed, if $y_n \in D(A^*)$ converge to y and $A^*y_n \to z$, then

$$(Ax, y) = \lim(Ax, y_n) = \lim(x, A^*y_n) = (x, z),$$

so that $y \in D(A^*)$ and $A^*y = z$].

A densely defined operator A is *symmetric* if

$$(Ax, y) = (x, Ay) \quad (x, y \in D(A)).$$

This is of course equivalent to the relation $A \subset A^*$. In particular, any symmetric operator is closable, and its closure $\overline{A}$ (which satisfies necessarily $\overline{A} \subset A^*$) is a closed symmetric operator. We say that A is *selfadjoint* if $A = A^*$, and *essentially selfadjoint* if $\overline{A} = A^*$. Note that we always have $A^* = (\overline{A})^*$ and $\overline{A} = A^{**}$, so that A is essentially selfadjoint if and only if $A^* = A^{**}$.

Let $[D(A^*)]$ denote the Hilbert space $D(A^*)$ with the *Hilbert* graph-norm, induced by the graph inner product

$$(x, y)_{A^*} := (x, y) + (A^*x, A^*y) \quad (x, y \in D(A^*)).$$

Since $A^* : [D(A^*)] \to X$ is continuous, the subspaces

$$D_+ := \ker(iI - A^*); \quad D_- := \ker(-iI - A^*) \tag{1}$$

are closed subspaces of $[D(A^*)]$; $A^*x = ix$ on D_+, $A^*y = -iy$ on D_-, and the subspaces are clearly orthogonal:

$$(x, y)_{A^*} = (x, y) + (ix, -iy) = 0$$

for all $x \in D_+$ and $y \in D_-$.

For x, y as above and $z \in D(\overline{A})$ for A *symmetric* (so that $\overline{A} \subset A^*$), we have

$$\begin{aligned}(z, x)_{A^*} &:= (z, x) + (A^*z, A^*x) = (z, x) + (\overline{A}z, ix)\\ &= (z, x) + (z, A^*ix) = (z, x) - (z, x) = 0,\end{aligned}$$

and similarly for y. Therefore the (Hilbert) direct sum of D_+ and D_- is contained in the orthocomplement Y of $D(\overline{A})$ in $[D(A^*)]$. On the other hand, if $u \in Y$, then for all $y \in D(A)$,

$$0 = (y, u)_{A^*} = (y, u) + (Ay, A^*u),$$

so that $(Ay, A^*u) = (y, -u)$; hence $A^*u \in D(A^*)$, and $A^*(A^*u) = -u$. Therefore $(iI - A^*)(-iI - A^*)u = 0$ (with commuting factors!), showing that

$$(-iI - A^*)u \in D_+; \quad (iI - A^*)u \in D_-$$

for all $u \in Y$. Since $u = (1/2i)[(iI - A^*)u - (-iI - A^*)u]$, we see that Y is contained in the direct sum of D_+ and D_-, hence equals it (by the preceding observation). Thus

$$[D(A^*)] = D(\overline{A}) \oplus D_+ \oplus D_-. \tag{2}$$

The subspaces D_+ and D_- are called the *deficiency subspaces* of A, and their Hilbert dimensions are the *deficiency indices* n_+ and n_-, respectively. Since it is generally true, for any densely defined operator T, that $\ker T^*$ equals the orthocomplement of $\operatorname{ran} T$, we have

$$D_+ = [\operatorname{ran}(-iI - A)]^\perp; \quad D_- = [\operatorname{ran}(iI - A)]^\perp, \tag{3}$$

where the orthocomplement is taken in the space X. In particular, we read from the decomposition (4) that the symmetric operator A is essentially selfadjoint if and only if both $iI - A$ and $-iI - A$ have dense range in X.

After this digression about symmetric operators, consider again a C_o-group of *unitary* operators $T(\cdot)$, and write its generator as $A = iH$. Then for all $x, y \in D(H) = D(A)$,

$$(Hx, y) = (-i) \lim_{t \to 0+} (t^{-1}[T(t) - I]x, y) = (-i) \lim(x, t^{-1}[T(-t) - I]y)$$
$$= -i(x, -Ay) = (x, Hy),$$

i.e., H is a (densely defined, closed) symmetric operator. Also $\sigma(H) = -i\sigma(A) \subset \mathbb{R}$ (see above!), so that, in particular, $iI - H$ and $-iI - H$ are *onto*. By the preceding remarks, this proves that H is selfadjoint. Let e^{itH} be the operator associated with H by means of the operational calculus for selfadjoint operators:

$$e^{itH}x := \int_{\mathbb{R}} e^{its} E(ds)x, \tag{4}$$

where $E(\cdot)$ is the resolution of the identity for H. A simple application of dominated convergence shows that e^{itH} is a C_o-group with generator $iH = A$. Therefore $T(t) = e^{itH}$. Since any C_o-semigroup of *unitary* operators extends in an obvious manner to a C_o-group, we have

Theorem 1.41 (Stone's Theorem). *Let $T(\cdot)$ be a C_o-(semi)group of unitary operators in Hilbert space. Then there exists a unique selfadjoint operator H such that $T(t) = e^{itH}$ for all t.*

Combining this with Theorem 1.40, and applying the uniqueness property of Fourier–Stieltjes transforms, we obtain

Corollary 1.42. *Let $T(\cdot)$ be a bounded C_o-group of operators in Hilbert space. Then there exists a unique (not necessarily selfadjoint) spectral measure $E(\cdot)$ on the Borel algebra of $\mathbb{R}$ such that*

$$T(t) = \int_{\mathbb{R}} e^{its} E(ds) \quad (t \in \mathbb{R}),$$

where the integral exists in the strong operator topology.

Using the terminology of [DS I–III], the generator of $T(\cdot)$ equals iS, where S is a *scalar-type spectral operator* with real spectrum, and $T(t) = e^{itS}$ (using the operational calculus for S). Also QSQ^{-1} is selfadjoint, with Q as in Theorem 1.40.

Remark. A neat way to deal with the Hilbert adjoint is to consider its graph $G(A^*) \subset X^2$, where the Cartesian product X^2 is a Hilbert space under the inner product

$$([x, y], [x', y']) := (x, x') + (y, y').$$

The operator

$$J : [x, y] \to [y, -x]$$

is a unitary operator on X^2, and a simple calculation shows that

$$G(A^*) = \{JG(A)\}^\perp.$$

One reads from this formula that A^* is closed (its graph is closed in X^2 as an orthocomplement!), and that $A^* = (\overline{A})^*$. Also, since J is unitary with $J^2 = I$,

$$\begin{aligned} G(A^{**}) &= [JG(A^*)]^\perp = JG(A^*)^\perp \\ &= J[JG(A)]^{\perp\perp} = J\,\overline{[JG(A)]} = J^2\,\overline{G(A)} = \overline{G(A)} = G(\overline{A}). \end{aligned}$$

Therefore $A^{**} = \overline{A}$.

A.19 Bochner's Theorem

Stone's theorem can be used to prove Bochner's theorem on the characterization of the Fourier–Stieltjes transforms of finite *positive* Borel measures.

Theorem 1.43 (Bochner's Theorem). *A continuous function $f : \mathbb{R} \to \mathbb{C}$ is the Fourier–Stieltjes transform of a finite positive Borel measure μ (i.e., $f(t) = \int_{\mathbb{R}} e^{its}\mu(ds)$) if and only if it is positive definite, that is,*

$$\sum_{j,k} f(t_j - t_k) c_j \overline{c_k} \geq 0$$

for all finite sequences $t_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$.

Proof. Let X_o be the complex vector space of all complex functions on $\mathbb{R}$ with *finitely* many nonzero values. Let

$$(\phi, \psi) = \sum_{t,s \in \mathbb{R}} f(t - s)\phi(t)\overline{\psi(s)},$$

where f is a given positive definite function, and $\phi, \psi \in X_o$. This is a *semi*-inner product on X_o (i.e., we may have $(\phi, \phi) = 0$ for nonzero ϕ). The set $K = \{\phi \in X_o; (\phi, \phi) = 0\}$ is a closed subspace of X_o. The factor space $X_1 = X_o/K$ is a pre-Hilbert space with the inner product $(\phi + K, \psi + K) = (\phi, \psi)$ (which is well-defined, i.e., independent of the choice of the cosets representatives). Let X be the completion of the pre-Hilbert space X_1. Define U_r^o

on X_o by $[U_r^o\phi](t) = \phi(t-r)$, $r \in \mathbb{R}$. Then $(U_r^o\phi, U_r^o\psi) = (\phi, \psi)$. Since U_r^o maps K into itself, it induces an operator U_r^1 of X_1 onto itself, well-defined by $U_r^1(\phi+K) = U_r^o\phi+K$, and U_r^1 preserves inner products. It extends uniquely to a unitary operator U_r of X onto itself. The family $\{U_r; r \in \mathbb{R}\}$ is a C_o-group on X (where the C_o-property follows from the assumed *continuity* of f). By Stone's theorem, $(U_r x, x) = \int_{\mathbb{R}} e^{irs}(E(ds)x, x)$ for all $x \in X$, where $(E(\cdot)x, x) = \|E(\cdot)x\|^2$ is a finite positive Borel measure. In particular, for $x_o = \phi_o + K$ with $\phi_o(t) = 1$ for $t = 0$ and vanishing otherwise, we have

$$(U_r x_o, x_o) = (U_r^1 x_o, x_o) = (U_r^o \phi_o, \phi_o) = f(r),$$

so that f is indeed the Fourier–Stieltjes transform of the finite positive Borel measure $(E(\cdot)x_o, x_o)$.

The converse is an easy calculation. □

It is also possible to deduce Stone's theorem from Bochner's. If $T(\cdot)$ is a C_o-group of unitary operators on the Hilbert space X, then $f := (T(\cdot)x, x)$ is continuous and positive definite (for each $x \in X$):

$$\sum_{j,k} f(t_j - t_k)c_j\overline{c_k} = \sum (T(t_k)^* T(t_j)x, x)c_j\overline{c_k}$$

$$= \sum (T(t_j)x, T(t_k)x)c_j\overline{c_k} = \left\| \sum_j c_j T(t_j)x \right\|^2 \geq 0.$$

Therefore, by Bochner's theorem, there exists a family $\{\mu(\cdot; x); x \in X\}$ of finite positive Borel measures on $\mathbb{R}$ such that

$$(T(t)x, x) = \int_{\mathbb{R}} e^{its}\mu(ds; x)$$

for all $t \in \mathbb{R}$ and $x \in X$.

Define

$$\mu(\cdot; x, y) := (1/4) \sum_{0 \leq k \leq 3} i^k \mu(\cdot; x + i^k y) \quad (x, y \in X).$$

These are (finite) complex Borel measures, and

$$(T(t)x, y) = \int_{\mathbb{R}} e^{its}\mu(ds; x, y) \quad (t \in \mathbb{R}; x, y \in X).$$

From this representation and the uniqueness property of the Fourier–Stieltjes transform, it is easy to deduce the existence of a resolution of the identity E such that $T(t) = \int e^{its}E(ds) = e^{itA}$ for the selfadjoint operator $A := \int sE(ds)$ with domain $D(A) = \{x \in X; \int s^2(E(ds)x, x) < \infty\}$. The details are given in an analogous proof in the next chapter, with X a *reflexive Banach space*.

B

The Semi-Simplicity Space for Groups

The main purpose of this section is to obtain a *Banach space version* of the spectral integral representation of bounded C_o-groups of operators in Hilbert space, which was deduced above either as a consequence of the theorems of Stone and Sz.-Nagy, or as an application of Bochner's characterization of the Fourier–Stieltjes transforms of finite positive Borel measures on $\mathbb{R}$. We shall construct a maximal Banach subspace of the given Banach space, with the property that the given group $T(\cdot)$ has a spectral integral representation on it. The construction of this so-called *semi-simplicity space* for $T(\cdot)$ is based on Bochner's characterization of the Fourier–Stieltjes transforms of *complex* Borel measures.

Theorem 1.44 (Bochner). *A function $f : \mathbb{R} \to \mathbb{C}$ is the Fourier–Stieltjes transform of a complex regular Borel measure μ with $\|\mu\| \le K$ if and only if it is continuous and*

$$\left| \sum_j c_j f(t_j) \right| \le K \left\| \sum_j c_j e^{it_j s} \right\|_\infty$$

for all finite sequences of real t_j and complex c_j.

Proof. See [[R2], p. 32]. □

B.1 The Bochner Norm

We consider the normed space $\mathbb{P} = \mathbb{P}(\mathbb{R})$ of all "trigonometric polynomials"

$$\phi(s) = \sum_j c_j e^{it_j s} \quad (c_j \in \mathbb{C}; t_j \in \mathbb{R}),$$

where the sum above is *finite*, with the supremum norm. Let Y be any Banach space. Given a function $f : \mathbb{R} \to Y$, we define the linear operator $B_f : \mathbb{P} \to Y$ by

S. Kantorovitz, *Topics in Operator Semigroups*,
Progress in Mathematics 281, DOI 10.1007/978-0-8176-4932-6_2,

$$B_f\phi := \sum_j c_j f(t_j),$$

for $\phi \in \mathbb{P}$ as above, and set

$$\|f\|_B := \|B_f\|,$$

where the norm on the right is the operator norm (that could be infinite a priori). We refer to $\|\cdot\|_B$ as the "Bochner norm," and consider the vector space

$$\mathbb{F}(Y) := \{f : \mathbb{R} \to Y; \|f\|_B < \infty\}.$$

Let $\phi_t(s) = e^{its}, \quad t, s \in \mathbb{R}$. Then for any $f : \mathbb{R} \to Y$,

$$\|f\|_B = \|B_f\| \geq \|B_f\phi_t\| = \|f(t)\| \quad (t \in \mathbb{R}),$$

so that

$$\|f\|_B \geq \|f\|_\infty := \sup_t \|f(t)\|.$$

Thus all functions in $\mathbb{F}(Y)$ are bounded, and $\|\cdot\|_B$ is a norm on that space. In addition, if $\{f_n\} \subset \mathbb{F}(Y)$ is $\|\cdot\|_B$-Cauchy, it converges uniformly in Y to some f. Given $\epsilon > 0$, let n_o be such that $\|f_n - f_m\|_B < \epsilon$ for all $n > n_o$. Then

$$\left\| \sum_j c_j [f_n(t_j) - f_m(t_j)] \right\| \leq \epsilon$$

for all c_j, t_j such that the corresponding ϕ has supremum norm equal to 1, and all $n, m > n_o$. Letting $m \to \infty$, we get

$$\left\| \sum_j c_j [f_n(t_j) - f(t_j)] \right\| \leq \epsilon$$

for all $n > n_o$, and all c_j, t_j as above. Thus $\|f_n - f\|_B \leq \epsilon < \infty$ (for $n > n_o$), i.e., $f_n - f \in \mathbb{F}(Y)$, and so $f = f_n - (f_n - f) \in \mathbb{F}(Y)$, and $f_n \to f$ in the Bochner norm. This shows that $\mathbb{F}(Y)$ is a Banach space with the Bochner norm.

A simple calculation shows that $\mathbb{F}(Y)$ contains all Fourier–Stieltjes transforms of Y-valued vector measures m (cf. [[DS I–III], Section IV.10]): if $f(t) := \int_{\mathbb{R}} e^{its} m(ds)$, then for all ϕ as above with $\|\phi\|_\infty = 1$,

$$\left\| \sum_j c_j f(t_j) \right\| = \left\| \int_{\mathbb{R}} \phi(s) m(ds) \right\| \leq \|m\|,$$

so that $\|f\|_B \leq \|m\| < \infty$, where $\|m\|$ denotes the "semi-variation" of m.

In particular, $\mathbb{F}(Y)$ contains the constant functions c, and $\|c\|_B = \|c\|$.

The Bochner norm is invariant under additive translation $f(t) \to f(t+c)$ and nonzero multiplicative translation $f(t) \to f(ct) \quad (c \in \mathbb{R})$.

If Y, Z are Banach spaces and $U \in B(Y,Z)$, then $U\mathbb{F}(Y) \subset \mathbb{F}(Z)$ and

$$\|Uf\|_B \leq \|U\| \, \|f\|_B \quad (f \in \mathbb{F}(Y)).$$

We have

$$\|f\|_B = \sup\{\|y^* f\|_B; y^* \in Y^*, \|y^*\| = 1\}.$$

By the Uniform Boundedness Theorem, $f \in \mathbb{F}(Y)$ if and only if $y^* f \in \mathbb{F}(\mathbb{C})$ for all $y^* \in Y^*$.

Also, if $F \to B(X,Y)$ for Banach spaces X, Y, then the following are equivalent:

(i) $F \in \mathbb{F}(B(X,Y))$;
(ii) $Fx \in \mathbb{F}(Y)$ for all $x \in X$;
(iii) $y^* Fx \in \mathbb{F}(\mathbb{C})$ for all $x \in X, y^* \in Y^*$.

In addition,

$$\begin{aligned}\|F\|_B &= \sup\{\|Fx\|_B; x \in X, \|x\| = 1\} \\ &= \sup\{\|y^* Fx\|_B; x \in X, y^* \in Y^*, \|x\| = \|y^*\| = 1\}.\end{aligned}$$

If $f \in \mathbb{F}(Y)$ is weakly *continuous*, then by Bochner's theorem, there corresponds to each $y^* \in Y^*$ a finite regular complex Borel measure $\mu(.; y^*)$ such that

$$y^* f(t) = \int_{\mathbb{R}} e^{its} \mu(ds; y^*) \quad (t \in \mathbb{R}),$$

and

$$\|\mu(.; y^*)\| \leq \|f\|_B \|y^*\|.$$

The uniqueness of the integral representation implies that for each $\delta \in \mathcal{B}(\mathbb{R})$ (the Borel algebra of $\mathbb{R}$), $\mu(\delta; .)$ is a linear functional on Y^* with norm $\leq \|f\|_B$. Therefore

$$\mu(\delta; y^*) = m(\delta) y^* \quad (y^* \in Y^*, \delta \in \mathcal{B}(\mathbb{R})),$$

where $m : \mathcal{B}(\mathbb{R}) \to Y^{**}$ is such that $m(\cdot)y^*$ is a regular finite Borel measure for each $y^* \in Y^*$. The element $\int_{\mathbb{R}} e^{its} m(ds) \in Y^{**}$, well-defined by the relation

$$\left[\int_{\mathbb{R}} e^{its} m(ds)\right] y^* = \int_{\mathbb{R}} e^{its} [m(ds) y^*] \quad (y^* \in Y^*),$$

coincides with the element $f(t) \in X$ (embedded in Y^{**} as usual). In this weakened sense, the weakly continuous elements of $\mathbb{F}(Y)$ are Fourier–Stieltjes transforms of Y^{**}-valued measures like m. When Y is reflexive, m is a weakly countably additive (hence strongly countably additive, by Pettis' theorem, cf. [DS I–III]) Y-valued measure, and we can write

$$f(t) = \int_{\mathbb{R}} e^{its} m(ds) \quad (t \in \mathbb{R}),$$

so that the weakly continuous functions in $\mathbb{F}(Y)$ are precisely the Fourier–Stieltjes transforms of vector measures m as above. We summarize the above discussion formally:

Proposition 1.45. *The space $\mathbb{F}(Y)$ is a Banach space for the Bochner norm, and contains all the Fourier–Stieltjes transforms of Y-valued measures with finite variation. Every weakly continuous function in the space is the Fourier–Stieltjes transform of a Y^{**}-valued measure m such that $m(\cdot)y^*$ is a regular complex Borel measure for each $y^* \in Y^*$. When Y is reflexive, every weakly continuous function in the space is the Fourier–Stieltjes transform of a strongly countably additive Y-valued measure m (such that $y^*m(\cdot)$ is a regular complex Borel measure for each y^*), and is in particular strongly continuous as well.*

Definition 1.46. *Suppose iA generates a C_o-group $T(\cdot)$ on the Banach space X. We set*

$$\|x\|_T = \|T(\cdot)x\|_B \quad (x \in X).$$

The semi-simplicity space for $T(\cdot)$ is the linear subspace of X

$$Z = Z_T = \{x \in X; \|x\|_T < \infty\},$$

with the norm $\|\cdot\|_T$.

Recall that a *Banach subspace of X* is a Banach space $(W, \|\cdot\|_W)$ such that $W \subset X$ and $\|w\|_W \geq \|w\|$ for all $w \in W$.

Lemma 1.47. *The semi-simplicity space for $T(\cdot)$, $(Z, \|\cdot\|_T)$, is a Banach subspace of X, invariant for any $U \in B(X)$ which commutes with $T(\cdot)$. Also $Z = X$ if and only if $\|T(\cdot)\|_B < \infty$ (in this case, the two norms on X are equivalent).*

Proof. Since $\|x\|_T \geq \|T(\cdot)x\|_\infty \geq \|x\|$, $(Z, \|\cdot\|_T)$ is a normed space. We prove completeness. If $\{x_n\}$ is Cauchy in $(Z, \|\cdot\|_T)$, it is also Cauchy in X; let x be its X-limit. Then $T(t)x_n \to T(t)x$ for each $t \in \mathbb{R}$. By definition of the $\|\cdot\|_T$-norm, $\{T(\cdot)x_n\}$ is Cauchy in the Banach space $(\mathbb{F}(X), \|\cdot\|_B)$. Therefore $T(\cdot)x_n \to f$ in that space, and since $\|\cdot\|_B \geq \|\cdot\|_\infty$, $f(t) = \lim_n T(t)x_n = T(t)x$ (limit in X), for each t. Thus $T(\cdot)x \in \mathbb{F}(X)$, i.e., $x \in Z$, and $\|x_n - x\|_T = \|T(\cdot)x_n - T(\cdot)x\|_B \to 0$, and Z (with the $\|\cdot\|_T$-norm) is a Banach subspace of X.

If $U \in B(X)$ commutes with $T(t)$ for each $t \in \mathbb{R}$, then for each $x \in Z$, we have $T(\cdot)x \in \mathbb{F}(X)$, and therefore $UT(\cdot)x \in \mathbb{F}(X)$ and $\|UT(\cdot)x\|_B \leq \|U\| \, \|T(\cdot)x\|_B$ (see above), which is equivalent in our present situation to $T(\cdot)[Ux] \in \mathbb{F}(X)$ (i.e., $Ux \in Z$) and $\|Ux\|_T \leq \|U\| \, \|x\|_T$. This shows that Z is U-invariant, and $\|U\|_{B(Z)} \leq \|U\|_{B(X)}$. If $\|T(\cdot)\|_B < \infty$, then for all trigonometric polynomials ϕ as above, and for all $x \in X$, $\|\sum_j c_j T(t_j)x\| \le \|\sum_j c_j T(t_j)\|\,\|x\| \le \|T(\cdot)\|_B\|x\|$, and therefore $\|x\|_T \le \|T(\cdot)\|_B\|x\| < \infty$, i.e., $Z = X$. Conversely, if $Z = X$, then for all $x \in X$, $\sup \|[\sum_j c_j T(t_j)]x\| < \infty$ (supremum over all ϕ as above), and therefore $\|T(\cdot)\|_B := \sup \|\sum_j c_j T(t_j)\| < \infty$ by the Uniform Boundedness Theorem. The equivalence of the norms is a consequence of the Closed Graph Theorem, or explicitly from the above discussion,

$$\|x\| \le \|x\|_T \le \|T(\cdot)\|_B\|x\| \quad (x \in X).$$

□

B.2 The Semi-Simplicity Space

We proceed now to prove a version of Stone's theorem for C_o-groups of operators on reflexive Banach spaces. The integral representation will be valid on the semi-simplicity space for the given group. First, we need a proper generalization of spectral measures.

Definition 1.48. *Let W be a Banach subspace of X.*

A spectral measure on W is a function

$$E(\cdot) : \mathcal{B}(\mathbb{R}) \to B(W),$$

such that

(i) $E(\mathbb{R}) = I$ *(the identity operator on W);*
(ii) *for each $x \in W$, $E(\cdot)x$ is a regular, strongly countably additive vector measure in X (with respect to the X-norm!); and*
(iii) $E(\delta \cap \epsilon) = E(\delta)E(\epsilon)$, *for all $\delta, \epsilon \in \mathcal{B}(\mathbb{R})$.*

Note that by [DS I–III, Corollary III.4.5], $E(\cdot)x$ is necessarily X-bounded for each $x \in W$.

Note also that (ii) is equivalent to

(ii′) for each $x \in W$ and $x^* \in X^*$, $x^*E(\cdot)x$ is a regular complex Borel measure.

This follows from Pettis' theorem (cf. [DS I–III]).

Let $\mathbb{B}(\mathbb{R})$ denote the Banach algebra of all bounded complex Borel functions on $\mathbb{R}$. For E as above and $h \in \mathbb{B}(\mathbb{R})$, the operator $\tau(h) : W \to X$ is defined by

$$\tau(h)x = \int_{\mathbb{R}} h(s)E(ds)x \quad (x \in W).$$

We then extend τ to the algebra $\mathbb{B}_{loc}(\mathbb{R})$ of all complex Borel functions on $\mathbb{R}$ that are bounded on each finite interval $[a, b]$, by letting

$$\tau(h)x = \lim_{a,b} \int_a^b h(s)E(ds)x := \int_{\mathbb{R}} h(s)E(ds)x$$

for $h \in \mathbb{B}_{loc}(\mathbb{R})$, where $\lim_{a,b}$ stands for the limit in X when $a \to -\infty$ and $b \to \infty$; the domain of $\tau(h)$ is the set of all $x \in W$ for which the limit exists.

We are now ready to state our Banach space version of the Stone Theorem.

Theorem 1.49. *Let $T(\cdot)$ be a C_o-group of operators on the reflexive Banach space X, with generator iA and $\sigma(A) \subset \mathbb{R}$. Let Z be the semi-simplicity space for $T(\cdot)$. Then there exists a spectral measure E on Z with the following properties:*

(a) $T(t)x = \int_{\mathbb{R}} e^{its} E(ds)x \quad (x \in Z; t \in \mathbb{R})$;
(b) *E commutes with every $U \in B(X)$ commuting with $T(\cdot)$;*
(c) *τ (corresponding to E) is a norm-decreasing algebra homomorphism of $\mathbb{B}(\mathbb{R})$ into $B(Z)$, such that $\tau(\phi_t) = T(t)|_Z$ for $\phi_t(s) = e^{its}, \quad s,t \in \mathbb{R}$;*
(d) *If $f_1(s) = s \quad (s \in \mathbb{R})$, then $A_Z = \tau(f_1)_Z$ (where the subscript Z means the part of the relevant operator in Z), that is,*
  (i) $D(A_Z) = \{x \in Z; \int_{\mathbb{R}} sE(ds)x$ *exists and belongs to* $Z\}$, *and*
  (ii) $Ax = \int_{\mathbb{R}} sE(ds)x \quad (x \in D(A_Z))$.

In addition, Z is maximal and E is unique in the following sense: if W is a Banach subspace of X and F is a spectral measure on W for which (c) is valid, then W is a Banach subspace of Z and $F(\cdot) = E(\cdot)|_W$.

Proof. For each $x \in Z$, the function $T(\cdot)x$ is a strongly continuous element of $\mathbb{F}(X)$. Since X is reflexive, Proposition 1.45 gives a strongly countably additive X-valued measure $m(.;x)$ on $\mathcal{B}(\mathbb{R})$ such that

$$T(t)x = \int_{\mathbb{R}} e^{its} m(ds; x) \tag{1}$$

for all $t \in \mathbb{R}, x \in Z$;

$$\|m(\cdot;x)\| \leq \|x\|_T; \tag{2}$$

and $x^* m(\cdot;x)$ is a regular complex Borel measure for each $x^* \in X^*$. The uniqueness property of the Fourier–Stieltjes transform of regular complex Borel measures and the linearity of the left-hand side of (1) imply that $m(\cdot;x) = E(\cdot)x$, where $E(\delta)$ is a linear transformation from Z to X, for each $\delta \in \mathcal{B}(\mathbb{R})$. By (1) with $t = 0$, $E(\mathbb{R}) = I|_Z$. We rewrite (1) in the form

$$T(t)x = \int_{\mathbb{R}} e^{its} E(ds)x \quad (t \in \mathbb{R}, x \in Z). \tag{1'}$$

If $U \in B(X)$ commutes with $T(\cdot)$ and $x \in Z$, then $Ux \in Z$ by Lemma 1.47, and by (1′),

$$\int e^{its} UE(ds)x = UT(t)x = T(t)Ux = \int e^{its} E(ds)Ux,$$

hence $UE(\delta)x = E(\delta)Ux$ for all $x \in Z, \delta \in \mathcal{B}(\mathbb{R})$.

For each $\phi, \psi \in \mathbb{P}$ with $\|\phi\|_\infty = \|\psi\|_\infty = 1$ and with respective parameters $c_j, t_j; c'_k, t'_k$ (notation as in the preceding subsection), we have for $x \in Z$,

$$\left\|\sum_k c'_k T(t'_k) \sum_j c_j T(t_j) x\right\| = \left\|\sum_{k,j} c'_k c_j T(t'_k + t_j) x\right\|$$

$$\leq \|x\|_T \left\|\sum_{k,j} c'_k c_j e^{is(t'_k + t_j)}\right\|_\infty$$

$$\leq \|x\|_T \|\phi\|_\infty \|\psi\|_\infty = \|x\|_T.$$

Therefore

$$\left\|\sum_j c_j T(t_j) x\right\|_T \leq \|x\|_T$$

for all parameters as above. Fix $h \in \mathbb{B}(\mathbb{R}), x \in Z$. Since $\sum_j c_j T(t_j)$ is a bounded operator commuting with $T(\cdot)$, we have by (2)

$$\left\|\sum_j c_j T(t_j) \tau(h) x\right\| = \left\|\tau(h) \sum_j c_j T(t_j) x\right\|$$

$$\leq \|h\|_\infty \left\|E(\cdot) \sum_j c_j T(t_j) x\right\|$$

$$\leq \|h\|_\infty \left\|\sum_j c_j T(t_j) x\right\|_T \leq \|h\|_\infty \|x\|_T.$$

Therefore

$$\|\tau(h) x\|_T \leq \|h\|_\infty \|x\|_T$$

for all $h \in \mathbb{B}(\mathbb{R}), x \in Z$. In particular

$$\tau : \mathbb{B}(\mathbb{R}) \to B(Z, \|\cdot\|_T) := B(Z)$$

has norm ≤ 1 ; actually, $\|\tau\| = 1$, because

$$\|\tau(\phi_r) x\|_T = \|T(r) x\|_T := \|T(\cdot) T(r) x\|_B$$

$$= \|T(\cdot + r) x\|_B = \|T(\cdot) x\|_B = \|x\|_T,$$

by the translation invariance of the Bochner norm on $\mathbb{F}(\mathbb{R})$.

Taking $h = \chi_\delta$ (the characteristic function of $\delta \in \mathcal{B}(\mathbb{R})$), we get that

$$\|E(\delta)\|_{B(Z)} \leq 1.$$

For $t, u \in \mathbb{R}$ and $x \in Z$, with all integrals below extending over $\mathbb{R}$, we have

$$\int e^{ius} E(ds)T(t)x = T(u)T(t)x = T(u+t)x$$
$$= \int e^{ius}[e^{its}E(ds)x].$$

By uniqueness for Fourier–Stieltjes transforms,

$$E(\delta)T(t)x = \int e^{its}\chi_\delta(s)E(ds)x$$

for all $\delta \in \mathcal{B}(\mathbb{R})$, etc. However, the left-hand side equals $T(t)E(\delta)x = \int e^{its}E(ds)E(\delta)x$ since $E(\delta)x \in Z$ for $x \in Z$. Therefore, again by uniqueness for Fourier–Stieltjes transforms,

$$E(ds)E(\delta)x = \chi_\delta E(ds)x, \tag{3}$$

so that

$$E(\sigma)E(\delta)x = \int \chi_\sigma \chi_\delta E(ds)x = E(\sigma \cap \delta)x$$

for all $\sigma, \delta \in \mathcal{B}(\mathbb{R})$ and $x \in Z$.

We conclude that E is a spectral measure on Z.

By (3),

$$\tau(h)\tau(\chi_\delta)x = \tau(h)E(\delta)x = \int h(s)\chi_\delta(s)E(ds)x = \tau(h\chi_\delta)x$$

for all $h \in \mathbb{B}(\mathbb{R}), \delta \in \mathcal{B}(\mathbb{R}), x \in Z$. By linearity of τ, it follows that $\tau(hg) = \tau(h)\tau(g)$ for all $h \in \mathbb{B}(\mathbb{R})$ and $g \in \mathbb{B}_o(\mathbb{R})$, the subalgebra of simple Borel functions. Next, for $g \in \mathbb{B}(\mathbb{R})$, choose simple Borel functions g_n converging uniformly to g. Then for all $x \in Z$,

$$\begin{aligned}\|\tau(hg)x - \tau(h)\tau(g)x\| &\le \|\tau[h(g-g_n)]x\| + \|\tau(h)\tau(g_n - g)x\| \\ &\le \|h(g-g_n)\|_\infty \|x\|_T + \|h\|_\infty \|\tau(g_n - g)x\|_T \\ &\le 2\|h\|_\infty \|g_n - g\|_\infty \|x\|_T \to 0\end{aligned}$$

as $n \to \infty$, and Statement (c) of the theorem is proved.

For all $t \in \mathbb{R}$, $\sigma(itA) = it\sigma(A) \subset i\mathbb{R}$, so that $R(t) := R(1; itA)$ is a well-defined bounded operator commuting with $T(\cdot)$. If $x \in R(t)Z$, say $x = R(t)z$ with $z \in Z$, then $x \in D(A) \cap Z$, and $Ax = (it)^{-1}(x - z) \in Z$ (for $t \neq 0$), i.e., $R(t)Z \subset D(A_Z)$. On the other hand, if $x \in D(A_Z)$, then $z := (1 - itA)x \in Z$, and therefore $x = R(t)z \in R(t)Z$. This shows that

$$D(A_Z) = R(t)Z \quad (0 \neq t \in \mathbb{R}). \tag{4}$$

Let $x \in D(A_Z)$; write then $x = R(t)z$ with $z \in Z$ and $0 \neq t \in \mathbb{R}$ fixed. By Theorem 1.15,

$$R(t)z = t^{-1}R(t^{-1}; iA)z = t^{-1}\int_0^\infty e^{-s/t}T(s)z\,ds$$
$$= \int_0^\infty e^{-u}T(tu)z\,du \tag{5}$$

for all $0 \neq t \in \mathbb{R}, z \in X$. For $z \in Z$, we get

$$R(t)z = \int_0^\infty e^{-u}\int_{\mathbb{R}} e^{itus}E(ds)z\,du$$
$$= \int_{\mathbb{R}}\int_0^\infty e^{-u(1-its)}\,duE(ds)z = \int (1-its)^{-1}E(ds)z, \tag{6}$$

where the interchange of the order of integration is justified by applying on both sides an arbitrary $x^* \in X^*$ and using Fubini's theorem.

For real $a < b$, we then have by (6) and the multiplicativity of τ on $\mathbb{B}(\mathbb{R})$ (for $x = R(t)z$ as above):

$$\int_a^b sE(ds)x = \int_a^b s(1-its)^{-1}E(ds)z \to \int_{\mathbb{R}} s(1-its)^{-1}E(ds)z$$

when $a \to -\infty$ and $b \to \infty$. Writing $s(1-its)^{-1} = it^{-1}[1-(1-its)^{-1}]$, the last integral is seen to equal

$$it^{-1}[z - R(t)z] = it^{-1}(z-x) = Ax \in Z.$$

This shows that $D(A_Z) \subset \{x \in Z; \int_{\mathbb{R}} sE(ds)x$ exists and belongs to $Z\}$, and $Ax = \int sE(ds)x$ on $D(A_Z)$.

On the other hand, if x belongs to the set on the right of (i) (in Statement (d) of the theorem), then denoting the integral in (i) by $z \in Z$, we obtain from the multiplicativity of τ, for any $t \neq 0$:

$$R(t)z = R(t)\lim_{a,b}\int_a^b sE(ds)x = \lim_{a,b} R(t)\int_a^b sE(ds)x$$
$$= \lim_{a,b}\int_a^b s(1-its)^{-1}E(ds)x$$
$$= \int_{\mathbb{R}} s(1-its)^{-1}E(ds)x = it^{-1}[x - R(t)x]$$

(cf. preceding calculation).

Therefore $x = R(t)(x - itz) \in R(t)Z = D(A_Z)$ by (4), and we conclude that Property (d) in the statement of the theorem is valid.

Finally, suppose $(W, \|\cdot\|_W)$ is a Banach subspace of X for which Property (c) (in the statement of the theorem) is valid, with $(W, \|\cdot\|_W)$ replacing $(Z, \|\cdot\|_T)$ and $\tau' : \mathbb{B}(\mathbb{R}) \to B(W, \|\cdot\|_W)$ (induced by the spectral measure F on W) replacing τ. Then for all $\phi \in \mathbb{P}$ with $\|\phi\|_\infty = 1$ and parameters c_j, t_j, we have

$$\left\| \sum_j c_j T(t_j) x \right\| = \|\tau'(\phi)x\| \leq \|\tau'(\phi)x\|_W \leq \|\tau'(\phi)\|_{B(W)} \|x\|_W \leq \|x\|_W$$

for all $x \in W$. Therefore $\|x\|_T \leq \|x\|_W$, and W is a Banach subspace of Z. Since $T(t)x = \int e^{its} F(ds)x = \int e^{its} E(ds)x$ for $x \in W$, the uniqueness property of the Fourier–Stieltjes transform implies that $F(\cdot)x = E(\cdot)x$ for all $x \in W$. □

Examples.

1. Let $T(\cdot)$ be the translation group on $L^p(\mathbb{R})$,

$$[T(t)x](s) = x(s-t) \quad (s, t \in \mathbb{R};\ x \in L^p(\mathbb{R})).$$

Suppose $1 < p < \infty$. One verifies easily (by using the Bochner and the Schoenberg criteria for Fourier–Stieltjes transforms of measures on $\mathbb{R}$, cf. Theorem 1.44 and Exercise 50) that for any bounded C_o-group $T(\cdot)$ on a reflexive Banach space X, and for any $x \in X$,

$$\|x\|_T = \sup \left\{ \left\| \int_{\mathbb{R}} f(t) T(t) x \, dt \right\| ;\ f \in L^1(\mathbb{R}), \|\hat{f}\|_\infty \leq 1 \right\}.$$

In our present example, we then have

$$\|x\|_T = \sup\{\|f * x\|_p ;\ f \in L^1,\ \|\hat{f}\|_\infty \leq 1\}.$$

This is precisely the norm $\|x\|_0$ considered in [Fi], and Z coincides then with the space $(L^p)_0$ discussed in this paper in the context of the multipliers problem for Fourier transforms. According to [Fi], we then know that Z is dense in L^p when $p > 2$, is the whole space when $p = 2$, and is equal to $\{0\}$ when $p < 2$.

2. Let $X = L^p([0,1])$ $(1 \leq p < \infty)$, and define the operators $T(t)$ by

$$[T(t)x](s) = e^{its} x(s) + it \int_s^1 e^{itu} x(u) \, du$$

for $t \in \mathbb{R}$, $s \in [0,1]$, and $x \in L^p([0,1])$. One verifies easily that $T(\cdot)$ is a uniformly continuous group of operators on $L^p([0,1])$, whose (bounded) generator is given by iA, with

$$[Ax](s) = s x(s) + \int_s^1 x(u) \, du$$

for $s \in [0,1]$ and $x \in L^p([0,1])$. (Cf. [K18].)

Let $BV([0,1])$ be the space of all functions of bounded variation in $[0,1]$. If $x \in BV([0,1])$, an integration by parts shows that

$$[T(t)x](s) = e^{it}x(1) - \int_s^1 e^{itu}\,dx(u) \quad (t \in \mathbb{R}),$$

which implies that $BV([0,1]) \subset Z$. In particular, Z is dense in X. However, $Z \neq X$, because A has real spectrum and $T(t) = e^{itA}$ is not uniformly bounded on $\mathbb{R}$, so that A cannot be a scalar-type spectral operator on X (cf. [K18]).

B.3 Scalar-Type Spectral Operators

We consider the special situation when the semi-simplicity space Z coincides with X. By Lemma 1.47, this happens if and only if $\|T(\cdot)\|_B < \infty$, and in this case the two norms $\|\cdot\|$ and $\|\cdot\|_T$ on X are equivalent. Let E be the spectral measure on $Z = X$ provided by Theorem 1.49. Since $\|E(\delta)x\|_T \leq \|x\|_T$ for all $x \in X$, the equivalence of the norms shows that $E : \mathcal{B}(\mathbb{R}) \to B(X)$ is a "spectral measure" in the usual sense, that is, an algebra homomorphism of the Boolean algebra $\mathcal{B}(\mathbb{R})$ into $B(X)$ such that $E(\cdot)x$ is regular and countably additive for each $x \in X$. Properties (i) and (ii) in Statement (d) become

(i) $D(A) = \{x \in X; \int_{\mathbb{R}} sE(ds)x := \lim_{a,b} \int_a^b sE(ds)x$ exists$\}$; and
(ii) $Ax = \int_{\mathbb{R}} sE(ds)x \quad (x \in D(A))$.

Using the terminology of [DS I–III], the operator A is *spectral of scalar type* (with real spectrum). The map τ defined above is now the usual operational calculus for the scalar-type spectral operator A, and in particular, the semigroup $T(\cdot)$ is precisely e^{itA}, as defined through this operational calculus. Note that when X is a Hilbert space, the condition $\|T(\cdot)\|_\infty < \infty$ was necessary and sufficient for the above conclusions (cf. Corollary 1.42), while our generalization to reflexive Banach space requires the stronger assumption $\|T(\cdot)\|_B < \infty$. This latter condition is, however, necessary as well, by Proposition 1.45 and Lemma 1.47.

We formalize the above discussion in

Corollary 1.50. *Let iA generate a C_o-group $T(\cdot)$ on the reflexive Banach space X. Then A is a scalar-type spectral operator with real spectrum if and only if $\|T(\cdot)\|_B < \infty$. In that case, $T(t) = e^{itA} := \int_{\mathbb{R}} e^{its}E(ds)$, where E is the resolution of the identity for A.*

Actually, we can restate this corollary *without assuming a priori that iA generates a C_o-group.* We need only to assume that A is densely defined, and has real spectrum. Let then

$$R(t) := (I - itA)^{-1} \quad (t \in \mathbb{R}).$$

We first establish some identities.

Theorem 1.51. *If iA (with $\sigma(A)$ real) generates a C_o-group $T(\cdot)$, then*

$$\|x\|_T = \sup_{n\geq 0} \|R^n x\|_B \quad (x \in X), \tag{1}$$

and

$$\|T(\cdot)\|_B = \sup_{n\geq 0} \|R^n\|_B.$$

Proof. By Theorem 1.36,

$$T(t)x = \lim_{n\to\infty} R^n(t/n)x \quad (x \in X; t \in R).$$

Therefore for each $\phi \in \mathbb{P}$ with $\|\phi\|_\infty = 1$ and parameters c_j, t_j,

$$\left\| \sum_j c_j T(t_j)x \right\| = \lim_n \left\| \sum_j c_j R^n(t_j/n)x \right\|.$$

Each norm on the right is $\leq \|R^n(\cdot/n)x\|_B = \|R^n x\|_B$, by the invariance of the Bochner norm under multiplicative translations, and this implies the inequality $\leq$ in (1).

On the other hand, by (5) in the proof of Theorem 1.38 (cf. proof of Lemma 5),

$$R(\lambda; A)^{(n-1)} = (-1)^{n-1}(n-1)! R(\lambda; A)^n \quad (\lambda \in \rho(A); n \geq 1).$$

The derivatives may be calculated by using Theorem 1.15, when A generates a C_o-semigroup $T(\cdot)$. We then obtain the following Laplace transform representation for the powers of the resolvent:

$$R(\lambda; A)^n x = \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x \, dt, \tag{2}$$

for all $x \in X, \Re\lambda > \omega$, and $n \geq 1$.

In our case, with the generators iA and $-iA$ of the semigroups $T(\cdot)$ and $S(t) := T(-t)(t \geq 0)$, respectively, a simple calculation leads to the formula

$$R^n(t)x = \frac{1}{(n-1)!} \int_0^\infty e^{-s} s^{n-1} T(ts)x \, ds, \tag{3}$$

for all $x \in X, n = 1, 2, \ldots$ and $t \in \mathbb{R}$.

Since $\|T(ts)x\|_B = \|T(t)x\|_B := \|x\|_T$ for each fixed $s > 0$, we have for all ϕ as above,

$$(n-1)! \left\| \sum_j c_j R^n(t_j)x \right\| = \left\| \int_0^\infty e^{-s} s^{n-1} \sum_j c_j T(t_j s)x \, ds \right\|$$
$$\leq \int_0^\infty e^{-s} s^{n-1} \|T(ts)x\|_B \, ds = (n-1)! \, \|x\|_T,$$

hence $\|R^n x\|_B \leq \|x\|_T$ for all $n \geq 0$, and (1) follows. The second identity is then an elementary consequence. □

We can restate now Corollary 1.50 without assuming a priori that iA is a generator.

Theorem 1.52. *Let A be a densely defined operator with real spectrum, acting in the reflexive Banach space X. Then A is a scalar-type spectral operator if and only if*

$$V_A := \sup_{n \geq 0} \|R^n\|_B < \infty$$

(in that case, iA generates the group e^{itA}, which is the Fourier–Stieltjes transform of the resolution of the identity for A).

Proof. If $V_A < \infty$, we surely have $\|R^n\|_\infty \leq V_A < \infty$ for all n.

Since

$$\lambda R(\lambda; iA) = R(1/\lambda) \quad (0 \neq \lambda \in \mathbb{R}),$$

we have

$$\|[\lambda R(\lambda; iA)]^n\| \leq V_A \quad (n \in \mathbb{N}; 0 \neq \lambda \in \mathbb{R}).$$

Also iA is densely defined (by hypothesis). By Theorem 1.39 (with $\omega - \omega' = 0$), iA generates a C_o-group $T(\cdot)$, and $\|T(\cdot)\|_B = V_A < \infty$ by Theorem 1.51. Therefore, by Corollary 1.50, A is a scalar-type spectral operator.

Conversely, if A is scalar-type spectral, let $E : \mathcal{B}(\mathbb{R}) \to B(X)$ be its resolution of the identity. Then iA generates the C_o-group

$$T(t) = e^{itA} := \int_{\mathbb{R}} e^{its} E(ds).$$

In particular, $\|T(\cdot)\|_B < \infty$ by Proposition 1.45, that is, $V_A < \infty$ (by Theorem 1.51). $\square$

C

Analyticity

A function $F : D \to B(X)$ (where D is a domain in $\mathbb{C}$) is *analytic* in D if

$$F'(z) := \lim_{h \to 0} h^{-1}[F(z+h) - F(z)]$$

exists in the uniform operator topology, for all $z \in D$. This is equivalent to the existence of that limit in the strong operator topology, and in the weak operator topology as well (cf. [HP], Theorem 3.10.1).

C.1 Analytic Semigroups

Definition 1.53. *The C_o-semigroup $T(\cdot)$ (on $[0, \infty)$) is analytic if it extends to an analytic function (also denoted $T(\cdot)$) in some sector*

$$S_\theta := \{z \in \mathbb{C}; |\arg z| < \theta, |z| > 0\},$$

$0 < \theta \le \pi$*, and* $\lim T(z)x = x$ *as* $z \to 0, z \in S_\theta$*, for all* $x \in X$*.*

The extended function necessarily satisfies the semigroup identity in S_θ:

$$T(z)T(w) = T(z+w) \quad (z, w \in S_\theta)$$

(cf. [HP], Theorem 17.2.2), and is also referred to as an *analytic semigroup in* S_θ.

The study of analyticity for C_o-semigroups can be restricted to uniformly bounded semigroups (consider $e^{-at}T(t)$ instead of $T(\cdot)$, for $a > \omega$). For simplicity, we consider only the special case of C_o-contraction semigroups, and the possibility of extending them as analytic semigroups of contractions in some sector. We refer to the literature for criteria applicable to the general case.

Theorem 1.54. *Let $T(\cdot)$ be a C_o-semigroup of contractions (on $[0, \infty)$), with generator A. Then $T(\cdot)$ extends to an analytic contraction semigroup in a sector S_θ ($0 < \theta \le \pi/2$) if and only if $e^{i\alpha}A$ generates a C_o-contraction semigroup for each $\alpha \in (-\theta, \theta)$.*

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Proof. *Necessity.* For each $\alpha \in (-\theta, \theta)$, define $T_\alpha(t) := T(te^{i\alpha})$, $t \geq 0$.

Clearly, $T_\alpha(\cdot)$ is a C_o-contraction semigroup. Denote its generator by A_α, and consider $\alpha > 0$ (the case $\alpha < 0$ is analogous). By Theorem 1.15, for all $s > 0$ and $x \in X$,

$$R(s; A_\alpha)x = \int_0^\infty e^{-st} T_\alpha(t)x \, dt$$
$$= e^{-i\alpha} \int_0^\infty \exp[-se^{-i\alpha} te^{i\alpha}] T(te^{i\alpha})x \, d(te^{i\alpha}). \tag{1}$$

The function $F(z) := \exp[-se^{-i\alpha} z] \, T(z)x$ is analytic in S_θ (for s, α, x fixed). Denote $C_a = \{z = ae^{i\phi}; 0 \leq \phi \leq \alpha\}$, oriented positively ($a > 0$). On C_a,

$$\|F(z)\| \leq \|x\| \, \exp[-s \, \Re(ze^{-i\alpha})] = \|x\| \, e^{-sa \, \cos(\alpha - \phi)} \leq \|x\| \, e^{-sa \, \cos \alpha}.$$

Therefore the integral of F over C_a has norm $\leq \pi a e^{-sa \, \cos \alpha} \|x\| \to 0$ when $a \to 0+$ and when $a \to \infty$, since $0 < \alpha < \theta \leq \pi/2$.

For $0 < a < b < \infty$, consider the closed contour

$$\Gamma_{a,b} := [a, b] + C_b - [a, b]e^{i\alpha} - C_a.$$

By Cauchy's theorem, $\int_{\Gamma_{a,b}} F(z) dz = 0$. Since the integrals of F on C_a and C_b converge strongly to 0 when $a \to 0+$ and $b \to \infty$, it follows that the right-hand side of (1) is equal to $e^{-i\alpha} \int_0^\infty \exp[-(se^{-i\alpha})t] \, T(t)x \, dt$. However, $\Re(se^{-i\alpha}) = s \, \cos \alpha > 0$, so that, by Theorem 1.15 (for the contraction case), the last expression is equal to $e^{-i\alpha} R(se^{-i\alpha}; A)x = R(s; e^{i\alpha} A)$. We conclude that A_α and $e^{i\alpha} A$ have equal resolvents on $\mathbb{R}^+$, and therefore $e^{i\alpha} A$ is indeed the generator of the C_o-contraction semigroup $T_\alpha(\cdot)$. □

Sufficiency. Suppose that for each $\alpha \in (-\theta, \theta)$, $e^{i\alpha} A$ generates a C_o-contraction semigroup $T_\alpha(\cdot)$. By Theorem 1.36,

$$T_\alpha(t)x = \lim_n \left[\frac{n}{t} R\left(\frac{n}{t}; e^{i\alpha} A\right)\right]^n x$$
$$= \lim_n \left[I - \frac{z}{n} A\right]^{-n} x, \tag{2}$$

where $z = te^{i\alpha}, t > 0$. Denote

$$F_n(z) := \left[I - \frac{z}{n} A\right]^{-n} = \left[\frac{n}{z} R\left(\frac{n}{z}; A\right)\right]^n.$$

Since A generates a C_o-contraction semigroup, F_n are analytic in $\Re \frac{n}{z} > 0$, hence in S_θ (if $z := te^{i\phi} \in S_\theta$, then $\Re(n/z) = (n/t) \cos \phi > 0$).

Since $F_n(z) = [\frac{n}{t} R(\frac{n}{t}; e^{i\alpha} A)]^n$, and $e^{i\alpha} A$ is the generator of a C_o-contraction semigroup, we have $\|F_n(z)\| \leq 1$ for all $z \in S_\theta$ (by Corollary 1.18).

For each $x \in X$ and $x^* \in X^*$, the sequence $\{x^* F_n(\cdot)x\}$ of complex analytic functions is uniformly bounded (by $\|x\| \, \|x^*\|$) in S_θ, hence is a normal family. It has then a subsequence converging uniformly on every compact subset of S_θ to a function $f(.; x, x^*)$ analytic in S_θ. By (2),

$$f(te^{i\alpha}; x, x^*) = x^* T_\alpha(t)x \tag{3}$$

for all $x \in X, x^* \in X^*, t > 0$, and $\alpha \in (-\theta, \theta)$.

Define $T(z) = T_\alpha(t)$, for $z = te^{i\alpha} \in S_\theta$. By (3), $T(\cdot)$ is analytic in S_θ. It coincides with the original semigroup on $[0, \infty)$ and is contraction-valued in the sector (by definition). It remains to verify that

$$\|T(z)x - x\| \to 0$$

as $z \to 0$, $z \in S_\theta$(for all $x \in X$). Since $\|T(\cdot) - I\| \leq 2$ in the sector, we may consider only x in the dense set $D(A) = D(e^{i\alpha}A)$. For such x, writing $z = te^{i\alpha} \in S_\theta$, we have (since $e^{i\alpha}A$ generates the C_o-contraction semigroup $T_\alpha(t) = T(te^{i\alpha})$)

$$\|T(z)x - x\| = \left\| \int_0^t T_\alpha(s) e^{i\alpha} Ax \, ds \right\| \leq t\|Ax\| = |z| \, \|Ax\|,$$

and the conclusion follows. □

C.2 The Generator of an Analytic Semigroup

We shall apply Theorem 1.54 to obtain various characterizations of generators of analytic semigroups.

Corollary 1. *Let A generate a C_o-semigroup of contractions $T(\cdot)$. Then $T(\cdot)$ extends as an analytic semigroup of contractions in a sector S_θ $(0 < \theta \leq \pi/2)$ if and only if*

$$\cos\alpha \, \Re(x^* Ax) - \sin\alpha \, \Im(x^* Ax) \leq 0 \tag{1}$$

for all unit vectors $x \in D(A)$ and $x^* \in X^*$ such that $x^* x = 1$, and for all $\alpha \in (-\theta, \theta)$.

Proof. For all $\alpha \in (-\theta, \theta)$, $e^{i\alpha}A$ is closed, densely defined, and for all $\lambda > 0$, $\lambda I - e^{i\alpha}A = e^{i\alpha}[\lambda e^{-i\alpha} I - A]$ is surjective, since $\Re(\lambda e^{-i\alpha}) = \lambda \cos\alpha > 0$ (cf. Theorem 1.26). Therefore, by Theorem 1.26, $e^{i\alpha}A$ generates a C_o-semigroup of contractions if and only if it is dissipative, i.e., if and only if

$$\Re(x^* e^{i\alpha} Ax) \leq 0$$

for all unit vectors $x \in D(A)$ and $x^* \in X^*$ such that $x^* x = 1$. This is precisely Condition (1), so that the corollary follows immediately from Theorem 1.54. □

When $\theta = \pi/2$ (i.e., for analytic semigroups in the right halfplane $\mathbb{C}^+$), we may consider "boundary values" on the imaginary axis.

Theorem 1.55. *Let $T(\cdot)$ be an analytic semigroup in $\mathbb{C}^+$, and suppose it is bounded in the "unit rectangle"*

$$Q := \{z = t + is \in \mathbb{C}; t \in (0, 1], s \in [-1, 1]\}.$$

Let $\nu := \log[\sup_Q \|T(\cdot)\|]$ (clearly $0 \le \nu < \infty$). Then for each $s \in \mathbb{R}$,

$$T(is) := \lim_{t \to 0+} T(t + is)$$

exists in $B(X)$ in the strong operator topology, and has the following properties (a)–(e):

(a) *$\{T(is); s \in \mathbb{R}\}$ is a C_o-group;*
(b) *$T(is)$ commutes with $T(z)$ for all $s \in \mathbb{R}, z \in \mathbb{C}^+$;*
(c) *$T(t + is) = T(t)T(is)$ for all $t > 0, s \in \mathbb{R}$;*
(d) *$T(\cdot)$ is of exponential type $\le \nu$ in the closed right halfplane, i.e.,*

$$\|T(z)\| \le K e^{\nu|z|} \quad (\Re z \ge 0);$$

(e) *If A is the generator of $\{T(t); t \ge 0\}$, then iA is the generator of the "boundary group" $\{T(is); s \in \mathbb{R}\}$.*

Proof. See Theorem 1.105 below (and its proof). □

Corollary 2. *Suppose that the generator A of the C_o-semigroup of contractions $T(\cdot)$ has real numerical range (i.e., $\nu(A) \subset \mathbb{R}$, cf. Definition 1.24). Then $T(\cdot)$ extends as an analytic semigroup of contractions in $\mathbb{C}^+$. In particular, the boundary group $\{T(is); s \in \mathbb{R}\}$ exists, and is a C_o-group of isometries (with generator iA).*

Proof. Condition (1) of Corollary 1 reduces here to $\cos\alpha\, \Re(x^* Ax) \le 0$ (for all parameters in their proper ranges), which is trivially satisfied (since $|\alpha| < \theta \le \pi/2$, and by Theorem 1.26 applied to A). Observe finally that a *group* of contractions consists in fact of isometries. □

Corollary 3. *Let A be a closed densely defined operator. Then the following statements are equivalent:*

(a) *A generates an analytic semigroup of contractions in the sector S_θ ($0 < \theta \le \pi/2$);*
(b) *$\|zR(z; A)\| \le 1$ for all $z \in S_\theta$;*
(c) *$zI - A$ is surjective for all $z \in S_\theta$, and $\Re[e^{i\alpha}\nu(A)] \le 0$ for all $\alpha \in (-\theta, \theta)$.*

Proof. Writing $z = te^{i\alpha}$, we see that Condition (b) is equivalent to

$$(\mathrm{b}')\ \|tR(t; e^{i\alpha}A)\| \le 1 \text{ for all } t > 0 \text{ and } \alpha \in (-\theta, \theta),$$

and (b′) is equivalent to (a), by Corollary 1.18 and Theorem 1.54.

Assume (c). For all $\alpha \in (-\theta, \theta)$, $e^{i\alpha}A$ is closed, densely defined, and for all $t > 0$, $tI - e^{i\alpha}A = e^{i\alpha}[te^{-i\alpha}I - A]$ is surjective. The inequality in (c) means that $e^{i\alpha}A$ is dissipative, and (a) follows from Theorems 1.26 and 1.54. Conversely, if (a) holds, then Theorems 1.26 and 1.54 imply that $e^{i\alpha}A$ is dissipative and $tI - e^{i\alpha}A$ is surjective for all $\alpha \in (-\theta, \theta)$, and this is equivalent to (c). □

Corollary 4. *Let A be a closed densely defined operator such that $zI - A$ is surjective for $\Re z > 0$ and $\nu(A) \subset (-\infty, 0]$. Then A generates an analytic semigroup of contractions $T(\cdot)$ in the right halfplane. In particular, the boundary group $\{T(is);\ s \in \mathbb{R}\}$ exists, and is a C_o-group of isometries (with generator iA).*

Proof. For *real* numerical range, the inequality in Corollary 3 (c) reduces to $\cos\alpha\, x^*Ax \le 0$ (for all parameters in their proper ranges), which is satisfied by hypothesis. The conclusion follows then from Corollaries 2 and 3. □

In case X is a *Hilbert* space, let $\pi : X^* \to X$ be the canonical isometric anti-isomorphism of X^* onto X given by the Riesz representation $x^*x = (x, \pi(x^*))$. Then

$$\nu(A) = \{(Ax, \pi(x^*)); x \in D(A), x^* \in X^*, \|x\| = \|x^*\| = (x, \pi(x^*)) = 1\}.$$

However, writing $\pi(x^*) = y$, we have (for x, x^* as in the above formula):

$$\|x - y\|^2 = \|x\|^2 - 2\Re(x, y) + \|y\|^2 = 1 - 2 + 1 = 0,$$

so that $y = x$, and

$$\nu(A) = \{(Ax, x); x \in D(A), \|x\| = 1\}.$$

Therefore A is *dissipative* if and only if

$$\Re(Ax, x) \le 0 \quad (x \in D(A)).$$

The inequality in Condition (c) of Corollary 3 becomes

$$\Re[e^{i\alpha}(Ax, x)] \le 0 \quad (x \in D(A)).$$

We then have

Corollary 5. *Let A generate a C_o-semigroup of contractions $T(\cdot)$ in Hilbert space. Then $T(\cdot)$ extends as an analytic semigroup of contractions in a sector S_θ $(0 < \theta \le \pi/2)$ if and only if*

$$\Re[e^{i\alpha}(Ax, x)] \le 0 \quad (x \in D(A), \alpha \in (-\theta, \theta)).$$

Corollary 6. *Let A be a closed densely defined operator in Hilbert space, such that $zI - A$ is surjective for $\Re z > 0$ and $(Ax, x) \le 0$ for all $x \in D(A)$. Then A generates an analytic semigroup of contractions in $\mathbb{C}^+$; the boundary group $\{T(is);\ s \in \mathbb{R}\}$ is a unitary C_o-group (with generator iA), and A is selfadjoint.*

Proof. By Corollary 4, A generates an analytic C_o-semigroup of contractions in $\mathbb{C}^+$. Since (Ax, x) is *real* for all $x \in D(A)$, A is symmetric. The boundary group $\{T(is); s \in \mathbb{R}\}$ in that corollary (with generator iA) satisfies (cf. Theorem 1.36):

$$\begin{aligned} T(is)x &= \lim_n \left[\frac{n}{s} R\left(\frac{n}{s}; iA\right)\right]^n x \\ &= \lim_n \left[\frac{n}{is} R\left(\frac{n}{is}; A\right)\right]^n x \end{aligned}$$

for all $s > 0$ and $x \in X$. Since $-iA$ is the generator of the C_o-semigroup $\{T(-is); s \ge 0\}$, we also have

$$\begin{aligned} T(-is)x &= \lim_n \left[\frac{n}{s} R\left(\frac{n}{s}; -iA\right)\right]^n x \\ &= \lim_n \left[\frac{n}{-is} R\left(\frac{n}{-is}; A\right)\right]^n x. \end{aligned}$$

Since A is symmetric, $R(z; A)^* = R(\overline{z}; A)$, and in particular, $R(z; A)$ is normal. Therefore, for all $x, y \in X$ and $s > 0$,

$$\begin{aligned} (T(-is)x, y) &= \lim_n \left(\left[\frac{n}{is} R\left(\frac{n}{is}; A\right)\right]^{n*} x, y\right) \\ &= \lim_n \left(x, \left[\frac{n}{is} R\left(\frac{n}{is}; A\right)\right]^n y\right) = (x, T(is)y), \end{aligned}$$

i.e.,

$$T(is)^* = T(-is) = T(is)^{-1}$$

for all $s > 0$ (hence for all $s \in \mathbb{R}$). Thus the boundary group is unitary; by Stone's theorem (Theorem 1.41), its generator iA has the form iH with H selfadjoint, that is, A *is selfadjoint.* □

A more direct way to prove the selfadjointness of A goes as follows. Suppose $y \in X$ satisfies $((iI - A)x, y) = 0$ for all $x \in D(A)$. Then $i(x, y) = (Ax, y)$ for all $x \in D(A)$. Take $x = R(s; A)y (\in D(A)!)$ for some $s > 0$. Then

$$i(R(s; A)y, y) = (AR(s; A)y, y) = ([sR(s; A) - I]y, y).$$

The left-hand side is pure imaginary, while the right-hand side is real (since the bounded operators appearing there are both selfadjoint). Therefore

$$(R(s; A)y, y) = s(R(s; A)y, y) - (y, y) = 0,$$

hence $(y, y) = 0$ and $y = 0$. This shows that $iI - A$ (and similarly, $-iI - A$) has dense range, which means that A is *essentially selfadjoint* (cf. "digression" preceding Theorem 1.41). Since A is closed, it is actually *selfadjoint*.

Note that the relation $T(t)x = \lim_n [\frac{n}{t} R(\frac{n}{t}; A)]^n x$ shows that the operators $T(t)$ are *selfadjoint* (for A symmetric). The longer discussion given above illustrates the "method of analytic continuation to the imaginary axis," which will be used later in the more general case of a *local semigroup of unbounded symmetric operators* to produce a selfadjoint operator H such that each $T(t)$ is a restriction of e^{-tH}.

D

The Semigroup as a Function of its Generator

In this section, we consider a C_o-semigroup $T(\cdot)$ as a function of its generator A, when A varies in the set of all generators of C_o-semigroups. The notation $T(\cdot) = T(\cdot; A)$ will be used to exhibit the generator A of the semigroup.

D.1 Noncommutative Taylor Formula

In order to get a feeling about a possible Taylor formula relating $T(\cdot; B)$ with $T(\cdot; A)$ and *derivatives* of the semigroup with respect to A (at the *point* A), we consider first the case of uniformly continuous semigroups (i.e., the variable generator varies in $B(X)$). This case can be formulated in an arbitrary complex Banach algebra $\mathcal{A}$ with identity I, and we may consider analytic functions on it, more general than the functions $f_t(A) := e^{tA}, t \geq 0, A \in \mathcal{A}$. As before, we denote the resolvent set of an element $S \in \mathcal{A}$ by $\rho(S)$, etc. We start with the following elementary

Lemma 1.56. *Let $S, T \in \mathcal{A}$ and $z \in \rho(S) \cap \rho(T)$. Then for all $n = 0, 1, 2, \ldots,$*

$$R(z;T) = \sum_{j=0}^{n} [R(z;S)(T-S)]^j R(z;S) + [R(z;S)(T-S)]^{n+1} R(z;T).$$

Proof. If $Q \in \mathcal{A}$ is such that $I - Q$ is invertible in $\mathcal{A}$, then one verifies directly the "geometric series addition formula"

$$(I-Q)^{-1} = \sum_{j=0}^{n} Q^j + Q^{n+1}(I-Q)^{-1}, \tag{*}$$

$n = 0, 1, 2, \ldots$. For $z \in \rho(S) \cap \rho(T)$, we take

$$\begin{aligned} Q := R(z;S)(T-S) &= R(z;S)[(zI-S)-(zI-T)] \\ &= I - R(z;S)(zI-T), \end{aligned}$$

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so that $I-Q = R(z;S)(zI-T)$ is indeed invertible in $\mathcal{A}$ with inverse equal to $R(z;T)(zI-S)$. Substituting in (*) and multiplying on the right by $R(z;S)$, the lemma follows. □

The formula of the lemma simplifies as follows when S, T are commuting elements of $\mathcal{A}$.

Lemma 1.57. *Let $S, T \in \mathcal{A}$ commute, and let $z \in \rho(S) \cap \rho(T)$. Then*

$$R(z;T) = \sum_{j=0}^{n} R(z;S)^{j+1}(T-S)^j + R(z;S)^{n+1}R(z;T)(T-S)^{n+1},$$

$n = 0, 1, 2, \dots$.

Given *arbitrary* elements $A, B \in \mathcal{A}$, we consider the *commuting* multiplication operators $L_A, R_B \in B(\mathcal{A})$ defined by

$$L_A U = AU; \quad R_B U = UB, \quad (U \in \mathcal{A}).$$

We then have

Lemma 1.58. *Let $A, B \in \mathcal{A}$ and $z \in \rho(A) \cap \rho(B)$. Then for all $n = 0, 1, 2, \dots,$*

$$R(z;B) = \sum_{j=0}^{n} R(z;A)^{j+1}(R_B - L_A)^j I + R(z;A)^{n+1}[(R_B - L_A)^{n+1} I]R(z;B)$$

and

$$R(z;B) = \sum_{j=0}^{n} [(L_B - R_A)^j I]R(z;A)^{j+1} + R(z;B)[(L_B - R_A)^{n+1} I]R(z;A)^{n+1}.$$

Proof. We apply Lemma 1.57 to the commuting elements $S = L_A$ and $T = R_B$ of the Banach algebra $B(\mathcal{A})$. If $z \in \rho(A) \cap \rho(B)$, then $z \in \rho(L_A) \cap \rho(R_B)$, $R(z;L_A) = L_{R(z;A)}$, and $R(z;R_B) = R_{R(z;B)}$. Therefore

$$R_{R(z;B)} = \sum_{j=0}^{n} L_{R(z;A)}^{j+1}(R_B - L_A)^j + L_{R(z;A)}^{n+1} R_{R(z;B)}(R_B - L_A)^{n+1}.$$

Applying this operator to the identity $I \in \mathcal{A}$, we obtain the first formula of the lemma. The second formula is deduced in the same manner, through the choice $S = R_A$ and $T = L_B$ in Lemma 1.57. □

The noncommutative Taylor formula for analytic functions on the Banach algebra $\mathcal{A}$ uses the Riesz–Dunford analytic operational calculus. Let f be a complex analytic function in an open neighborhood Ω of the spectrum $\sigma(B)$ of $B \in \mathcal{A}$. If $K \subset \Omega$ is compact, we denote by $\Gamma(K, \Omega)$ any finite union of positively oriented simple closed Jordan contours in Ω, that contains K in its interior. The element $f(B) \in \mathcal{A}$ is defined by

$$f(B) = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z; B) dz,$$

where $\Gamma = \Gamma(\sigma(B), \Omega)$, and the definition is independent of the choice of such Γ (cf. [DS I–III]).

Theorem 1.59. *Let $\mathcal{A}$ be a complex Banach algebra with identity I. Let $A, B \in \mathcal{A}$, and let f be a complex function analytic in a neighborhood Ω of $\sigma(A) \cup \sigma(B)$. Then for $n = 0, 1, 2, \ldots$,*

$$f(B) = \sum_{j=0}^{n} \frac{f^{(j)}(A)}{j!} (R_B - L_A)^j I + L_n(f, A, B)$$

and

$$f(B) = \sum_{j=0}^{n} (L_B - R_A)^j I \, \frac{f^{(j)}(A)}{j!} + R_n(f, A, B),$$

where the "left" and "right" remainders L_n and R_n are given by the formulas

$$L_n = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z; A)^{n+1} (R_B - L_A)^{n+1} I \; R(z; B) \, dz$$

and

$$R_n = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z; B) (L_B - R_A)^{n+1} I \; R(z; A)^{n+1} \, dz,$$

with $\Gamma = \Gamma(\sigma(A) \cup \sigma(B), \Omega)$.

Proof. For Γ as above, the Riesz–Dunford operational calculus satisfies

$$f^{(j)}(A) = \frac{j!}{2\pi i} \int_{\Gamma} f(z) R(z; A)^{j+1} dz,$$

and the theorem follows from Lemma 1.58 by integration. □

Note that

$$(R_B - L_A)^j I = \sum_{k=0}^{j} \binom{j}{k} (-A)^k B^{j-k},$$

with a similar formula for $(L_B - R_A)^j I$.

When A, B *commute*, these formulas reduce to $(B - A)^j$, and the Taylor formula of the theorem reduces to its "classical" form

$$f(B) = \sum_{j=0}^{n} \frac{f^{(j)}(A)}{j!} (B - A)^j$$
$$+ \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z; A)^{n+1} R(z; B) \, dz \; (B - A)^{n+1}.$$

When f is analytic in a "large" disk, the "Taylor formula" of Theorem 1.59 implies a noncommutative Taylor series expansion:

Theorem 1.60. *Let $\mathcal{A}$ be a complex Banach algebra with identity I, and let $A, B \in \mathcal{A}$. Suppose f is a complex function analytic on the closed disk*

$$\{z \in \mathbb{C}; |z| \leq 2\|A\| + \|B\|\}.$$

Then

$$f(B) = \sum_{j=0}^{\infty} \frac{f^{(j)}(A)}{j!}(R_B - L_A)^j I$$
$$= \sum_{j=0}^{\infty}(L_B - R_A)^j I.\frac{f^{(j)}(A)}{j!},$$

with both series converging strongly in $\mathcal{A}$.

Proof. It suffices to prove that the remainders L_n and R_n converge strongly to 0 in $\mathcal{A}$.

Fix $r > 2\|A\|+\|B\|$ such that C_r(:= the positively oriented circle of radius r centered at 0) and its interior are contained in the domain of analyticity Ω of f. Clearly $\sigma(A) \cup \sigma(B)$ lies in the interior of C_r, so we can take $\Gamma = C_r$ in Theorem 1.59. Let $M_r := \max_{z \in C_r} |f(z)|$.

On C_r, we have $\|R(z; A)\| = \|\sum_{n=0}^{\infty} \frac{A^n}{z^{n+1}}\| \leq (r - \|A\|)^{-1}$, and similarly for $R(z; B)$. Therefore

$$\|L_n\| \leq \frac{rM_r}{(r - \|A\|)^{n+1}(r - \|B\|)}\|(R_B - L_A)^{n+1}I\|,$$

with a similar estimate for R_n (replace the last factor by $\|(L_B - R_A)^{n+1}I\|$).

However,

$$\|(R_B - L_A)^{n+1}I\| = \left\|\sum_{j=0}^{n+1}\binom{n+1}{j}(-A)^j B^{n+1-j}\right\| \leq (\|A\| + \|B\|)^{n+1},$$

and similarly for the letters R, L interchanged. Therefore

$$\|L_n\| \leq \frac{rM_r}{r - \|B\|}\left[\frac{\|A\| + \|B\|}{r - \|A\|}\right]^{n+1} \to 0$$

as $n \to \infty$, because $\|A\| + \|B\| < r - \|A\|$, and similarly for R_n. □

Taking $f(z) = f_t(z) := e^{tz}$ (for $t \geq 0$ fixed) in the "Taylor formula" of Theorem 1.59, we obtain

$$e^{tB} = e^{tA} \sum_{j=0}^{n} \frac{t^j}{j!}(R_B - L_A)^j I + L_n, \tag{1}$$

with the appropriate expression for the remainder L_n.

This is the "noncommutative Taylor formula" we wish to generalize to the case of strongly continuous semigroups.

For (generally) unbounded operators A, B, we use the (suggestive) notation

$$(B-A)^{[j]} := (R_B - L_A)^j I := \sum_{k=0}^{j} \binom{j}{k} (-A)^k B^{j-k}$$

with maximal domain

$$D((B-A)^{[j]}) = \bigcap_{k=0}^{j} D(A^k B^{j-k}).$$

The dense $T(\cdot;A)$-invariant core for A consisting of all the C^∞-vectors for A (cf. Theorem 1.8) is denoted by $D^\infty(A)$. The *type* of $T(\cdot;A)$ is $\omega(A)$ (cf. Section A.3). We can state now

Theorem 1.61. *Let A, B be generators of C_o-semigroups such that $D^\infty(B) \subset D^\infty(A)$. Fix $a > \max[\omega(A), \omega(B)]$. Then for $n = 0, 1, 2, \ldots$ and $c > a$,*

$$T(t;B)x = T(t;A) \sum_{j=0}^{n} \frac{t^j}{j!} (B-A)^{[j]} x + L_n(t;A,B)x$$

for all $x \in D^\infty(B)$ and $t \geq 0$, where the "n-th remainder" L_n is given by

$$L_n(t;A,B)x = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{tz} R(z;A)^{n+1} (B-A)^{[n+1]} R(z;B)x \, dz;$$

the integral converges strongly in X as a "Cauchy Principal Value" and is independent of $c > a$.

Proof. Let A_s and B_u be the Hille–Yosida approximations of A and B, respectively ($s, u > a$; cf. Lemma 1.16 and paragraph following Corollary 1.18).

We recall that there exists $M > 0$ and $r > a$ such that

$$\|e^{tA_s}\| \leq Me^{at}; \quad \|e^{tB_u}\| \leq Me^{at} \tag{2}$$

for all $s, u > r$ and $t \geq 0$. In particular, A_s and B_u have their spectra in the closed halfplane $\{z \in \mathbb{C}; \Re z \leq a\}$, and

$$\|R(z;A_s)\| \leq \frac{M}{\Re z - a} \quad (\Re z > a), \tag{3}$$

with a similar estimate for $R(z;B_u)$ (for all $s, u > r$).

Recall that, in the strong operator topology,

$$e^{tA_s} \to T(t;A) \quad (t \geq 0) \tag{4}$$

and

$$R(z;A_s) \to R(z;A) \quad (\Re z > a) \tag{5}$$

as $s \to \infty$. Also, for all $x \in D(A)$,

$$A_s x \to Ax \tag{6}$$

as $s \to \infty$ (cf. Lemma 1.16, the proof of Theorem 1.17, and Theorem 1.32).

By (3), it follows from (5) that for all $j \in \mathbb{N}$ and $\Re z > a$,

$$R(z; A_s)^j \to R(z; A)^j \tag{7}$$

in the strong operator topology, as $s \to \infty$.

By (6), in the strong operator topology,

$$\lim_{s \to \infty} A_s R(z; A) = AR(z; A), \tag{8}$$

for each $z \in \rho(A)$.

By (3) and the definition of A_s,

$$\|A_s R(z; A)\| = \|sAR(s; A)R(z; A)\| = s\|R(s; A)[AR(z; A)]\|$$

$$\leq s\,\|R(s; A)\|\,\|AR(z; A)\| \leq \frac{Ms}{s-a}\|AR(z; A)\| \leq 2M\,\|AR(z; A)\|$$

for all $s > a$. This uniform boundedness together with (8) imply that for all $m \in \mathbb{N}$,

$$\lim_{s \to \infty} [A_s R(z; A)]^m = [AR(z; A)]^m \tag{8'}$$

in the strong operator topology. Since

$$D(A^m) = R(z; A)^m X \quad (m \in \mathbb{N}; z \in \rho(A)),$$

writing $x \in D(A^m)$ in the form $x = R(z; A)^m y$ for a suitable $y \in X$, we obtain (since A_s commutes with $R(z; A)$):

$$A_s^m x = A_s^m R(z; A)^m y = [A_s R(z; A)]^m y \to [AR(z; A)]^m y = A^m x.$$

Thus for all $m \in \mathbb{N}$,

$$\lim_{s \to \infty} A_s^m x = A^m x \quad (x \in D(A^m)). \tag{9}$$

For $0 \leq k \leq j$, $x \in D(B^{j-k})$, and $s > r$ fixed, it follows from (9) that

$$(-A_s)^k B_u^{j-k} x \to (-A_s)^k B^{j-k} x$$

(as $u \to \infty$). Therefore

$$\lim_{u \to \infty} (B_u - A_s)^{[j]} x = (B - A_s)^{[j]} x \tag{10}$$

for all $x \in D(B^j)$. Hence

$$\lim_{u \to \infty} R(z; A_s)^{j+1} (B_u - A_s)^{[j]} x = R(z; A_s)^{j+1} (B - A_s)^{[j]} x \tag{11}$$

for all $x \in D(B^j)$, $\Re z > a$, $s > r$, and $j = 0, 1, \ldots$.

If $x \in D((B-A)^{[j]})$, then for $0 \le k \le j$, $B^{j-k}x \in D(A^k)$, and therefore (9) implies that $(-A_s)^k B^{j-k}x \to (-A)^k B^{j-k}x$ as $s \to \infty$, hence

$$\lim_{s\to\infty} (B-A_s)^{[j]}x = (B-A)^{[j]}x. \tag{12}$$

Together with (3) and (7), this implies that

$$\lim_{s\to\infty} R(z;A_s)^m (B-A_s)^{[j]}x = R(z;A)^m (B-A)^{[j]}x \tag{13}$$

for all $x \in D(B-A)^{[j]})$, $m \in \mathbb{N}$, and $\Re z > a$.

If $x \in \bigcap_{j=0}^n D((B-A)^{[j]})$, then surely $x \in D(B^n)$, and it follows from (11) and (13) that

$$\lim_{s\to\infty}\lim_{u\to\infty}\sum_{j=0}^n R(z;A_s)^{j+1}(B_u-A_s)^{[j]}x = \sum_{j=0}^n R(z;A)^{j+1}(B-A)^{[j]}x \tag{14}$$

for $\Re z > a$.

On the other hand, by Lemma 1.58, for all $x \in X$, the left-hand side of (14) is equal to

$$R(z;B_u)x - R(z;A_s)^{n+1}(B_u-A_s)^{[n+1]}R(z;B_u)x. \tag{15}$$

If $x \in D(B^{m-1})(= R(z;B)^{m-1}X)$ (for any $m \in \mathbb{N}$), then $x = R(z;B)^{m-1}y$ for a suitable $y \in X$, and therefore

$$B_u^m R(z;B_u)x = [B_u R(z;B_u)][B_u R(z;B)]^{m-1}y.$$

The operators in the first bracket on the right are equal to $zR(z;B_u) - I$, and are therefore uniformly bounded (with respect to u) by $1 + \frac{M\,|z|}{\Re z - a}$ (by (3)), and converge (as $u \to \infty$) to $zR(z;B) - I = BR(z;B)$ (by (5)) in the strong operator topology. The second bracket converges to $[BR(z;B)]^{m-1}y$, by (8′) for B. It follows that

$$\begin{aligned} B_u^m R(z;B_u)x &\to [BR(z;B)][BR(z;B)]^{m-1}y \\ &= B^m R(z;B)^m y = B^m R(z;B)x \end{aligned}$$

for all $x \in D(B^{m-1})$ and $\Re z > a$.

Therefore, for $s > r$ fixed and $x \in D(B^n)$, the right-hand side of (15), which is equal to

$$R(z;B_u)x - \sum_{k=0}^{n+1}\binom{n+1}{k} R(z;A_s)^{n+1}(-A_s)^k B_u^{n+1-k}R(z;B_u)x,$$

converges as $u \to \infty$ to

$$R(z;B)x - R(z;A_s)^{n+1}(B-A_s)^{[n+1]}R(z;B)x$$

(cf. (5)).

If $x \in D(B^n)$ is such that $R(z;B)x \in D((B-A)^{[n+1]})$, it follows from (13) that the last expression converges to

$$R(z;B)x - R(z;A)^{n+1}(B-A)^{[n+1]}R(z;B)x$$

as $s \to \infty$.

We then conclude from (14) that the following generalization of Lemma 1.58 (first formula) is valid:

Lemma 1. *Let A, B be generators of C_o-semigroups. Then for $\Re z >$* $\max[\omega(A), \omega(B)]$,

$$R(z;B)x = \sum_{j=0}^{n} R(z;A)^{j+1}(B-A)^{[j]}x + R(z;A)^{n+1}(B-A)^{[n+1]}R(z;B)x$$

for all $x \in \bigcap_{j=0}^{n} D((B-A)^{[j]})$ such that $R(z;B)x \in D((B-A)^{[n+1]})$ (i.e., for all x in the maximal domain of the right-hand side).

Assume now that $D^\infty(B) \subset D^\infty(A)$. If $0 \le k \le j$ and $x \in D^\infty(B)$, then $B^{j-k}x \in D^\infty(B) \subset D^\infty(A) \subset D(A^k)$, so that $x \in D((-A)^k B^{j-k})$. Hence $x \in \bigcap_{j=0}^{n} D((B-A)^{[j]})$ for all n. Since also $R(z;B)x \in D^\infty(B)$, we have $R(z;B)x \in D((B-A)^{[n+1]})$ as well, and the formula in the lemma is valid for all $x \in D^\infty(B)$.

We need the following generalization of the second formula in Theorem 1.15.

Lemma 2. *Let A generate the C_o-semigroup $T(\cdot;A)$. Then for $t > 0$, $c > \omega(A)$, and $j = 0, 1, 2, \ldots$,*

$$\lim_{\tau\to\infty} \frac{1}{2\pi i} \int_{c-i\tau}^{c+i\tau} e^{tz} R(z;A)^{j+1}x\, dz = \frac{t^j}{j!} T(t;A)x \quad (x \in D(A)).$$

Proof of Lemma 2. Since $R(z;A)^{j+1} = \frac{(-1)^j}{j!} R(z;A)^{(j)}$, we may integrate by parts j times to show that the integral appearing in the lemma is equal to

$$\left[-e^{tz} \sum_{k=0}^{j-1} \frac{(j-k-1)!}{j!} t^k R(z;A)^{j-k}x\right]_{c-i\tau}^{c+i\tau} + \frac{t^j}{j!} \int_{c-i\tau}^{c+i\tau} e^{tz} R(z;A)x\, dz. \quad (16)$$

The "integrated part" has norm

$$\le e^{ct} \sum_{k=0}^{j-1} t^k (\|R(c+i\tau;A)^{j-k}x\| + \|R(c-i\tau;A)^{j-k}x\|).$$

If $x \in D(A)$, write $x = R(\lambda; A)y$ for some λ with $\Re\lambda > a$. Then since $j-k \geq 1$,

$$\|R(c+i\tau; A)^{j-k}x\| = \left\| R(c+i\tau; A)^{j-k-1} \frac{R(\lambda; A)y - R(c+i\tau; A)y}{c+i\tau-\lambda} \right\|$$

$$\leq \frac{M^{j-k}\|y\|}{(c-a)^{j-k-1}|c+i\tau-\lambda|} \left(\frac{1}{c-a} + \frac{1}{\Re\lambda - a} \right) \to 0$$

as $\tau \to \infty$, and similarly for $c - i\tau$. Therefore the integrated part in (16) converges to 0 when $\tau \to \infty$. By Theorem 1.15, the integral in (16) converges to $2\pi i T(t; A)x$ (for $x \in D(A)$), and the lemma follows.

If $x \in D^\infty(B)$ and $t > 0$, we have by Lemma 1

$$\int_{c-i\tau}^{c+i\tau} e^{tz} R(z; A)^{n+1} (B-A)^{[n+1]} R(z; B)x \, dz$$

$$= \int_{c-i\tau}^{c+i\tau} e^{tz} R(z; B)x \, dz - \sum_{j=0}^{n} \int_{c-i\tau}^{c+i\tau} e^{tz} R(z; A)^{j+1} (B-A)^{[j]} x \, dz. \quad (17)$$

However, for $x \in D^\infty(B)$, we surely have $x \in D(B)$, so that the first term on the right-hand side converges to $2\pi i T(t; B)x$, by Theorem 1.15 (when $\tau \to \infty$). We observed above that $B^{j-k}x \in D^\infty(A)$ for all $0 \leq k \leq j$, and therefore $(-A)^k B^{j-k} x \in D^\infty(A)$, and so $(B-A)^{[j]}x \in D^\infty(A) \subset D(A)$. Hence, by Lemma 2, the sum on the right-hand side of (17) converges to

$$2\pi i T(t; A) \sum_{j=0}^{n} \frac{t^j}{j!} (B-A)^{[j]} x. \quad (18)$$

This shows that the remainder L_n in Theorem 1.61 converges (as a "Cauchy Principal Value"), its "value" is independent of $c > a$, and is equal to

$$T(t; B) - T(t; A) \sum_{j=0}^{n} \frac{t^j}{j!} (B-A)^{[j]} x. \qquad \square$$

D.2 Analytic Families of Semigroups

In the previous section, we considered a semigroup as a function of its generator. Suppose now that the generator depends analytically (in some sense) on a complex parameter. We shall prove presently that the corresponding semigroup depends analytically on the parameter (in the usual sense).

Let Ω be a region in $\mathbb{C}$ (that is, an open, connected, nonempty subset of $\mathbb{C}$). For each $z \in \Omega$, let $A(z)$ be the generator of the C_o-semigroup $T(\cdot; z)$ on the Banach space X. In general, the constants of exponential growth a

and M (such that $\|T(t;z)\| \le M\,e^{at}$ for all $t \ge 0$) depend on z. The following standing *uniformity hypothesis* is made:

Uniformity hypothesis. For each compact subset K of Ω, there exist constants $a = a(K) \ge 0$ and $M = M(K) \ge 1$ such that

$$\|T(t;z)\| \le M\,e^{at} \tag{1}$$

for all $t > 0$ and $z \in K$.

We shall be concerned with the following "analyticity properties" of the family of operators $\{A(z);\ z \in \Omega\}$:

Definition 1.62.

1. *Generator analyticity: for all x in a common domain D for all $A(z)$ ($z \in \Omega$), $A(\cdot)x$ is analytic in Ω.*
2. *Resolvent analyticity: for each subregion $\Delta \subset\subset \Omega$ (i.e., Δ has compact closure $\overline{\Delta}$ contained in Ω), the resolvent $R(\lambda; A(\cdot))$ is analytic in Δ for all $\lambda > a'$ (for some constant $a' = a'(\overline{\Delta}) \ge 0$).*
3. *Semigroup analyticity: $T(t; A(\cdot))$ is analytic in Ω for each $t > 0$.*

We shall clarify the relations between the above analyticity properties. The following elementary lemma, which is contained implicitly in the proof of Theorem 1.54, will be used repeatedly.

Lemma 1.63. *Let $a < b \le \infty$. Let $\{x_s(\cdot);\ s \in (a,b)\}$ be a family of X-valued functions, analytic in Ω and uniformly bounded on compact subsets of Ω, such that $x_s(\cdot) \to x(\cdot)$ weakly, pointwise in Ω, as $s \to b$. Then $x(\cdot)$ is analytic in Ω.*

Proof. For each $x^* \in X^*$, the family of complex analytic functions $\{x^* x_s(\cdot);\ s \in (a,b)\}$ is uniformly bounded on compact subsets of Ω, and is therefore normal (cf. [R1, Theorem 14.6]). There exists therefore a sequence $\{x^* x_{s_k}(\cdot)\}$ converging uniformly on compact subsets of Ω to a function f analytic in Ω (cf. [R1, Theorem 10.27]). Since $x^* x_s(\cdot) \to x^* x(\cdot)$ pointwise in Ω (as $s \to b$), it follows that $x^* x(\cdot) = f$. Hence $x^* x(\cdot)$ is analytic in Ω for each $x^* \in X^*$, and therefore $x(\cdot)$ is analytic in Ω by Theorem 3.10.1 in [HP]. □

Theorem 1.64. *Semigroup analyticity and resolvent analyticity are equivalent.*

Proof.

1. Assume *semigroup analyticity.* Let $\Delta \subset\subset \Omega$, set $K := \overline{\Delta}$, and let $a = a(K)$ and $M = M(K)$ as in (1). By (1), $R(\lambda;\ A(z))$ exists for $\lambda > a$, and is given by the absolutely convergent Laplace transform

$$R(\lambda;\ A(z))x = \int_0^\infty e^{-\lambda t} T(t;z)x\,dt, \tag{2}$$

for all $\lambda > a$, $z \in \Delta$, and $x \in X$.

For $z.w \in \Delta$, $\lambda > a$, and $x \in X$,

$$\|R(\lambda;\, A(z))x - R(\lambda;\, A(w))x\| \leq \int_0^\infty e^{-\lambda t}\|T(t;z)x - T(t;w)x\|\,dt. \quad (3)$$

Since $T(t;\cdot)$ is analytic, hence continuous in Ω, the integrand in (3) converges pointwise to zero as $z \to w$. It is also majorized by

$$2M\,\|x\|\,e^{-(\lambda-a)t} \in L^1(0,\infty).$$

By the Dominated Convergence Theorem, it follows that $R(\lambda;\, A(\cdot))x$ is continuous in Δ (for $\lambda > a$).

If Γ is a triangular path lying together with its interior in Δ, then by Fubini's theorem, for all $\lambda > a$ and $x \in X$,

$$\int_\Gamma R(\lambda;\, A(z))x\,dz = \int_0^\infty e^{-\lambda t}\int_\Gamma T(t;z)x\,dz\,dt = 0$$

(by Cauchy's theorem, since $T(t;\cdot)x$ is analytic in Ω). By Morera's theorem, it follows that $R(\lambda;\, A(\cdot))x$ is analytic in Δ, for all $\lambda > a$ and $x \in X$. By Theorem 3.10.1 in [HP], this proves *resolvent analyticity* of the family of generators.

2. Assume *resolvent analyticity.* Let $\Delta \subset\subset \Omega$, and let a, M be as in (1). Set $B(\cdot) := A(\cdot) - aI$. For each $z \in \Delta$, $B(z)$ generates the C_o-semigroup $S(t;z) := e^{-at}T(t;z)$ $(t \geq 0)$, which satisfies

$$\|S(t;z)\| \leq M \quad (t \geq 0,\, z \in \Delta). \quad (4)$$

Hence (cf. Theorem 1.17 with $a = 0$)

$$\|[\lambda R(\lambda;\, B(z))]^n\| \leq M \quad (\lambda > 0,\, z \in \Delta,\, n \in \mathbb{N}). \quad (5)$$

Fix $t > 0$ and $x \in X$. The family of X-valued functions

$$x_n(\cdot) := [(n/t)R(n/t;\, B(\cdot))]^n x \quad (n \in \mathbb{N})$$

is uniformy bounded in Δ by $M\,\|x\|$ (by (5)). Since $R(\lambda;\, B(\cdot)) = R(\lambda + a;\, A(\cdot))$. the functions $x_n(\cdot)$ are analytic in Δ for all $n > n_0$ (for some $n_0 \in \mathbb{N}$), by the "resolvent analyticity" assumption, and converge pointwise to $S(t;\cdot)x$ (strongly in X), by Theorem 1.36. By Lemma 1.63, it follows that $S(t;\cdot)x$ is analytic in Δ. Therefore $T(t;\cdot)x = e^{at}S(t;\cdot)x$ is analytic in Δ, for all $x \in X$ and $t > 0$. By Theorem 3.10.1 in [HP], we conclude that $T(t;\cdot)$ is analytic in Ω for each $t > 0$. □

Definition 1.65. *The common domain D of $\{A(z);\, z \in \Omega\}$ is resolvent-invariant if for each compact $K \subset \Omega$, there exists a constant $a'' = a''(K) \geq 0$ such that D is $R(\lambda;\, A(z))$-invariant for all $\lambda > a''$ and $z \in K$.*

Theorem 1.66.

1. *Assume "semigroup analyticity." Then "generator analyticity" holds iff* $A(\cdot)x$ *is uniformly bounded on compact subsets of* Ω*, for each* x *in a common domain* D.
2. *Conversely, "generator analyticity" on a dense resolvent-invariant common domain implies "semigroup analyticity."*

Proof.

1. Since "generator analyticity" implies trivially uniform boundedness of $A(\cdot)x$ on compact subsets of Ω (for each $x \in D$), we assume "semigroup analyticity" and the said uniform boundedness, and we prove "generator analyticity." Fix $x \in D$ and $\delta > 0$, and set

$$x_t(x) := t^{-1}[T(t;z)x - x] \quad (0 < t < \delta). \tag{6}$$

Each function $x_t(\cdot)$ is analytic in Ω. If $K \subset \Omega$ is compact and a, M are as in (1), then for all $z \in K$,

$$\begin{aligned}\|x_t(z)\| &= \|t^{-1}\int_0^t T(s;z)A(z)x\,ds\| \le t^{-1}\int_0^t M\,e^{as}\,ds\,\|A(z)x\| \\ &\le M\,e^{a\delta}\sup_{z\in K}\|A(z)x\|.\end{aligned}$$

Thus $\{x_t(\cdot);\, 0 < t < \delta\}$ is uniformly bounded on compact subsets of Ω. Also $x_t(\cdot) \to A(\cdot)x$ strongly, pointwise in Ω, as $t \to 0+$. By Lemma 1.63, $A(\cdot)x$ is analytic in Ω.
2. Assume "generator analyticity" on the dense resolvent-invariant common domain D. Let $\Delta \subset\subset \Omega$ be a subregion, set $K = \overline{\Delta}$, and let a, M be the corresponding constants as in (1). *Fix* $z \in \Delta$, $x \in D$ and $\lambda > \max(a, a'')$. Then $y := R(\lambda;\, A(z))x \in D$, and for all $w \in \Delta$,

$$R(\lambda; A(z))x - R(\lambda; A(w))x = R(\lambda; A(w))[A(z) - A(w)]y, \tag{7}$$

hence

$$\|R(\lambda; A(z))x - R(\lambda; A(w))x\| \le M(\lambda - a)^{-1}\|A(z)y - A(w)y\| \to 0$$

as $w \to z$, by continuity of the analytic function $A(\cdot)y$ in Ω. Thus $R(\lambda; A(\cdot))x$ is continuous in Δ for each $x \in D$. Since D is dense in X and $\|R(\lambda; A(w))\| \le M(\lambda - a)^{-1}$ for all $w \in \Delta$, it follows that $R(\lambda; A(\cdot))x$ is continuous in Δ for all $x \in X$.
With notation as before, set $v := [A(\cdot)y]'(z)$. For all $w \in \Delta$, $w \neq z$, we have by (7)

$$\|(z-w)^{-1}[R(\lambda;A(z))x - R(\lambda;A(w))x] - R(\lambda;A(z))v\|$$
$$\leq \|R(\lambda;A(w))\,\{(z-w)^{-1}[A(z)y - A(w)y] - v\}\|$$
$$+ \|R(\lambda;A(w))v - R(\lambda;A(z))v\|.$$

The first summand above is

$$\leq M(\lambda-a)^{-1}\|(z-w)^{-1}[A(z)y - A(w)y] - [A(\cdot)y]'(z)\| \to 0$$

as $w \to z$, while the second summand $\to 0$ (as $w \to z$), by the continuity of $R(\lambda;A(\cdot))v$ at z (proved above). Thus $R(\lambda;A(\cdot))x$ is analytic in Δ for each $x \in D$ and $\lambda > a' := \max(a, a'')$.

For $x \in X$ arbitrary, we use the density of D to get a sequence $\{x_n\} \subset D$ converging strongly to x. Let $r = \sup_n \|x_n\|$. Then

$$\|R(\lambda;A(\cdot))x_n\| \leq M\,r(\lambda-a)^{-1}$$

on Δ, and for each $\lambda > a'$, the analytic functions $R(\lambda;A(\cdot))x_n$ converge to $R(\lambda;A(\cdot))x$ strongly, pointwise in Δ. By Lemma 1.63, it follows that $R(\lambda;A(\cdot))x$ is analytic in Δ for each $\lambda > a'$. Thus "resolvent analyticity" holds, and this is equivalent to the desired "semigroup analyticity," by Theorem 1.64. □

We consider an application to Theorem 1.66, which has an evident relevance to differential equations.

Let A be the generator of a C_o-semigroup $T(\cdot)$ on the Banach space X. Let $\Omega \subset \mathbb{C}$ be a region, and suppose $\{B(z);\, z \in \Omega\}$ is a family of *closed* operators in X satisfying the hypothesis

(H1) $T(t)X \subset D(B(z))$ for all $z \in \Omega$ and $t > 0$).

It follows from (H1) that $B(z)T(\cdot)$ is a strongly continuous $B(X)$-valued function and $\|B(z)T(\cdot)\|$ is a measurable function on $(0,\infty)$ for each fixed $z \in \Omega$. (Cf. comments preceding Theorem 1.38.)

Also, if ω is the type of $T(\cdot)$, then for each $\epsilon > 0$, $a > \omega$, and $z \in \Omega$, there exist a constant $M = M(\epsilon, a, z) \geq 1$ such that $\|B(z)T(t)\| \leq M(\epsilon, a, z)e^{at}$ for all $t \geq \epsilon$ (cf. discussion preceding Theorem 1.38).

In order to control the growth of $\|B(z)T(\cdot)\|$ *on the whole ray* $[0,\infty)$, *uniformly on* Ω, we make the following second hypothesis:

(H2) For each compact $K \subset \Omega$, there exist constants $a = a(K) \geq 0$ and $M = M(K) > 0$, and a positive function $h = h_K \in L^1(0,1)$, such that

$$\|B(z)T(t)\| \leq Me^{at} \quad (z \in K;\, t \geq 1)$$

and

$$\|B(z)T(t)\| \leq h(t) \quad (z \in K;\, 0 < t < 1).$$

For each fixed $z \in \Omega$, $B(z)$ satisfies the hypothesis of Theorem 1.38. Therefore, by Lemma 1 following Theorem 1.38,

$$D(A) \subset D(B(z)) \quad (z \in \Omega), \tag{8}$$

and by Theorem 1.38, the operator

$$A(z) := A + B(z), \tag{9}$$

with domain $D := D(A)$, generates a C_o-semigroup $T(\cdot; z)$. Our last hypothesis on the family $\{B(z);\, z \in \Omega\}$ is

(H3) $B(\cdot)x$ is analytic in Ω for each $x \in D$.

Theorem 1.67. *Let $A(\cdot)$ be defined as in (9), where A and $B(\cdot)$ satisfy hypotheses (H1)–(H3). Then "semigroup analyticity" holds for the family of semigroups $\{T(\cdot; z);\, z \in \Omega\}$ generated by the operators $A(z)$, $z \in \Omega$.*

In particular, the solution $u(\cdot; z)$ of the Abstract Cauchy Problem

$$\frac{\partial u(t; z)}{\partial t} = A(z)u(t; z) \quad u(0; z) = x, \quad z \in \Omega$$

is analytic in Ω with respect to the parameter z, for each $t \geq 0$.

Proof. Let $K \subset \Omega$, and let a, M, h be the parameters associated with K as in (H2). For $r > a$, set

$$q(r; z) := \int_0^\infty e^{-rt} \|B(z)T(t)\|\, dt. \tag{10}$$

(The integral makes sense, since the integrand is a positive measurable function.)

We break the integral into two integrals, over $(0, 1)$ and $(1, \infty)$, respectively. By (H2), we obtain the estimate

$$q(r; z) \leq Q(r) := \int_0^1 e^{-rt} h(t)\, dt + M\,(r - a)^{-1} e^{-(r-a)}, \tag{11}$$

for all $z \in K$ and $r > a$, where $Q(r)$ is clearly *independent* of z. By the Dominated Convergence Theorem, $Q(r) \to 0$ when $r \to \infty$. We can then *fix* $r = r(K) > a$ such that $Q(r) < 1$. For this r, $q(r; z) < 1$ for all $z \in K$, and it follows from the proof of Theorem 1.38 (cf. the estimate for $\|S(\cdot)\|$, which is $\|T(\cdot; z)\|$ in the present notation) that for all $z \in K$

$$\|T(t; z)\| \leq \frac{M}{1 - q(r; z)} e^{rt} \leq M' e^{rt}, \tag{12}$$

where

$$M' := \frac{M}{1 - Q(r)}$$

depends only on K (and is necessarily ≥ 1). Thus the *uniformity hypothesis* (1) is satisfied by the family of C_o-semigroups $\{T(\cdot; z);\, z \in \Omega\}$.

The common domain $D := D(A)$ of the operators $A(z)$, $z \in \Omega$, is dense in X, as the domain of a C_o-semigroup generator (cf. Theorem 1.2). For each compact $K \subset \Omega$, with notation as above, it follows from Lemma 1 (following Theorem 1.38) that for all $z \in K$, $\lambda > r$, and $x \in X$,

$$\|B(z)R(\lambda; A)x\| = \left\| \int_0^\infty e^{-\lambda t} B(z)T(t)x\, dt \right\| \leq q(r; z)\, \|x\| \leq Q(r)\, \|x\|.$$

Since $Q(r) < 1$, this implies that the series

$$R(\lambda; A) \sum_{n=0}^{\infty} [B(z)R(\lambda; A)]^n$$

converges in $B(X)$ when $\lambda > r$, and its sum is equal to $R(\lambda;\, A(z))$ (cf. proof of Lemma 2, following Theorem 1.38). In particular,

$$R(\lambda;\, A(z))X \subset R(\lambda; A)X = D$$

for all $z \in K$ and $\lambda > r$. This means that D is "resolvent-invariant" (cf. Definition 1.65). By (H3), $A(\cdot)x$ is clearly analytic in Ω for each $x \in D$. The conclusion of the theorem follows now from Theorem 1.66, Part 2. □

An important special case of Theorem 1.67 is the following

Corollary. *Let A be the generator of a C_o-semigroup $T(\cdot)$, and let $B(\cdot) : \Omega \to B(X)$ be analytic in the region $\Omega \subset \mathbb{C}$. Then "semigroup analyticity" holds for the family of C_o-semigroups $\{T(\cdot; z);\, z \in \Omega\}$ generated by the operators $A(z) := A + B(z)$, $z \in \Omega$.*

Proof. We need only to verify that (H2) holds in the present case. By Theorem 1.1, there exist constants $a \geq 0$ and $M_1 \geq 1$ such that $\|T(t)\| \leq M_1 e^{at}$ for all $t \geq 0$. For each compact $K \subset \Omega$ and $x \in X$, by continuity of the analytic function $B(\cdot)x$ on Ω, we have $\sup_{z \in K} \|B(z)x\| < \infty$. Therefore, by the Uniform Boundedness Theorem,

$$M_2(K) := \sup_{z \in K} \|B(z)\| < \infty.$$

Hence

$$\|B(z)T(t)\| \leq M_1 M_2(K)\, e^{at} \quad (t \geq 0),$$

so that (H2) is satisfied. □

We consider now two families of *contraction* C_o-semigroups $\{S(\cdot; z);\, z \in \Omega\}$ and $\{T(\cdot; z);\, z \in \Omega\}$, where Ω is a region in $\mathbb{C}$. Suppose the generators $A(z)$ and $B(z)$ of $S(\cdot; z)$ and $T(\cdot; z)$ respectively satisfy the following conditions for each $z \in \Omega$:

(1) $D(A(z)) \subset D(B(z))$; and
(2) there exist constants $a(z) \in [0,1)$ and $b(z) \geq 0$ such that

$$\|B(z)x\| \leq a(z)\,\|A(z)x\| + b(z)\,\|x\|$$

for all $x \in D(A(z))$.

For each $z \in \Omega$, $B(z)$ is dissipative and $A(z)$-bounded with $A(z)$-bound < 1 (cf. Theorem 1.26 and Definition 1.28). By Theorem 1.30, it follows that $C(z) := A(z) + B(z)$ generates a contraction C_o-semigroup $U(\cdot; z)$, which is given explicitly by the Trotter Product Formula (cf. Theorem 1.37):

$$U(t;z) = \lim_n [S(t/n;z)T(t/n;z)]^n \tag{13}$$

in the strong operator topology.

Corollary 1.68. *Suppose "semigroup analyticity" holds for the families $\{S(\cdot;z);\, z \in \Omega\}$ and $\{T(\cdot;z);\, z \in \Omega\}$ of contraction C_o-semigroups, whose respective generators $A(z)$ and $B(z)$ satisfy Conditions (1) and (2). Then "semigroup analyticity" holds for the family of contraction C_o-semigroups $\{U(\cdot;z);\, z \in \Omega\}$ generated by the operators $C(z) = A(z) + B(z)$.*

Proof. Observe that the product of $B(X)$-valued analytic functions in Ω is analytic in Ω. Therefore, for each *fixed* $t \geq 0$, $x \in X$, and $n \in \mathbb{N}$, the function

$$x_n(\cdot) := [S(t/n;\cdot)T(t/n;\cdot)]^n x$$

is analytic in Ω. We have on Ω:

$$\|x_n(\cdot)\| \leq \|x\|$$

and

$$\lim_n x_n(\cdot) = U(t;\cdot)x$$

pointwise, strongly in X (by (13)). By Lemma 1.63 and Theorem 3.10.1 in [HP], it follows that $U(t;\cdot)$ is analytic in Ω for each $t \geq 0$. □

E

Large Parameter

In this section, we shall obtain some results on the asymptotic behavior of a C_o-semigroup $T(\cdot)$. We consider first the simple cases of analytic semigroups and of semigroup averages. The more delicate study of stability for general C_o-semigroups concludes this section.

E.1 Analytic Semigroups

We begin with the case of an *analytic* semigroup. Thus, $T(\cdot)$ is assumed to have an analytic extension to some sector

$$S_\theta := \{z \in \mathbb{C};\ |z| > 0,\ |\arg z| < \theta\},$$

where $0 < \theta \leq \pi$.

(As before, we use the same notation $T(\cdot)$ for the extension. Necessarily, the semigroup property extends to the sector, i.e.,

$$T(z + w) = T(z)T(w) \quad (z, w \in S_\theta).$$

The extension is assumed to be strongly continuous at 0, and is also referred to as an "analytic semigroup.")

Let A be the generator of $T(\cdot)$, and let $D^\infty(A)$ be its space of C^∞-vectors. By Theorem 1.8 and its proof,

$$D^\infty(A) = \bigcap_{n=1}^{\infty} D(A^n). \tag{1}$$

The subspace $D^\infty(A)$ is dense and $T(z)$-invariant for all $z \in S_\theta$, and for all $n \in \mathbb{N}$,

$$[T(\cdot)x]^{(n)} = A^n T(\cdot)x = T(\cdot)A^n x \tag{2}$$

on S_θ. (The extension to complex variable is straightforward.)

S. Kantorovitz, *Topics in Operator Semigroups*,
Progress in Mathematics 281, DOI 10.1007/978-0-8176-4932-6_5,

Theorem 1.69. *Let* $T(\cdot)$ *be a* C_o*-semigroup, analytic in a sector* S_θ*, such that* $\|T(\cdot)\|$ *is bounded in every proper subsector. Let* A *be its generator, and* $0 < \alpha < \theta$*. Denote*

$$M = M(\alpha) := \sup\{\|T(z)\|;\, |z| > 0,\, |\arg z| \le (\alpha+\theta)/2\},$$

and

$$\delta = \delta(\alpha) := \sin((\theta-\alpha)/2).$$

Then

$$\|A^n T(z)x\| \le \frac{M\, n!}{(\delta|z|)^n}\, \|x\| \tag{3}$$

for all $n \in \mathbb{N}$*,* $x \in D^\infty(A)$*, and* $z \in S_\alpha$*.*

Proof. Fix $n \in \mathbb{N}$, $z = a\, e^{i\phi} \in S_\alpha$, and $x \in D^\infty(A)$. Let Γ be the positively oriented circle centered at z with radius δa. By the choice of δ, the circle Γ and its interior are contained in the set

$$\{w \in \mathbb{C};\, |w| > 0,\, |\arg w| \le (\alpha+\theta)/2\} \subset S_\theta.$$

By (2) and Cauchy's formula for the n-th derivative of the analytic function $T(\cdot)x$ in S_θ,

$$A^n T(z)x = \frac{n!}{2\pi i} \int_\Gamma \frac{T(w)x}{(w-z)^{n+1}}\, dw. \tag{4}$$

The integrand in (4) has norm $\le M\,\|x\|/(\delta a)^{n+1}$ for all $w \in \Gamma$, and (3) follows then from (4). □

Corollary 1.70. *Let* $T(\cdot)$ *be as in Theorem 1.69. Then (with notation as in the theorem)*

$$\|zAT(z)\| \le M/\delta \tag{5}$$

for all $z \in S_\alpha$*.*

Proof. For each $z \in S_\alpha$ $(0 < \alpha < \theta)$, the operator $AT(z)$ is in $B(X)$. Since $D^\infty(A)$ is dense in X, (5) follows from the case $n = 1$ in Theorem 1.69. □

The concept of an *analytic vector* will play a role in Section C of Part II. An analytic vector for the arbitrary operator A is a vector $x \in D^\infty(A) := \bigcap_n D(A^n)$ such that

$$\sum_n (t^n/n!)\|A^n x\| < \infty \tag{6}$$

for some $t > 0$. An immediate consequence of Theorem 1.69 is the following

Corollary 1.71. *Let* $T(\cdot)$ *be as in Theorem 1.69, and let*

$$D := \{T(z)x;\, z \in S_\theta,\, x \in D^\infty(A)\}.$$

Then D *is a set of analytic vectors for* A*.*

Proof. Let $z \in S_\theta$. Choose $0 < \alpha < \theta$ such that $z \in S_\alpha$, and let M and δ be associated with α as in Theorem 1.69. Pick $0 < t < \delta|z|$. For all $x \in D^\infty(A)$ and $n \in \mathbb{N}$, we have by (3)

$$(t^n/n!)\|A^n T(z)x\| \leq M\,\|x\|\,(t/\delta|z|)^n,$$

which clearly implies that $T(z)x$ is an analytic vector for A. □

Corollary 1.72. *Let $T(\cdot)$ be as in Theorem 1.69. Then its generator A has a dense set of analytic vectors.*

Proof. Let $\epsilon > 0$ and $y \in X$. Since $D^\infty(A)$ is dense in X (cf. Theorem 1.8), there exists $x \in D^\infty(A)$ such that $\|y - x\| < \epsilon/2$. By the C_o-property of $T(\cdot)$, there exists $h > 0$ such that $\|x - T(h)x\| < \epsilon/2$. Then $\|y - T(h)x\| < \epsilon$, and $T(h)x$ is an analytic vector for A, by Corollary 1.71. □

Corollary 1.73. *Let $T(\cdot)$ be as in Theorem 1.69. Then the following statements are equivalent:*

1. $x \in \overline{range(A)}$.
2. $T(z)x \to 0$ *strongly as* $z \to \infty$ *within any proper subsector of* S_θ.
3. *There exists a sequence* $\{z_k\}$ *diverging to* ∞ *within a proper subsector of* S_θ, *such that* $T(z_k)x \to 0$ *weakly.*

Proof. Since 2 implies 3 trivially, we need only to prove that 1 implies 2 and 3 implies 1.

1 implies 2. Let $\epsilon > 0$, $0 < \alpha < \theta$, and $x \in \overline{range(A)}$. Let then $y \in D(A)$ be such that $\|x - Ay\| < \epsilon/(2M)$. With notation as in Theorem 1.69, for all $z \in S_\alpha$ such that $|z| > (2M\,\|y\|)/(\delta\epsilon)$, we have

$$\begin{aligned}\|T(z)x\| &\leq \|T(z)Ay\| + \|T(z)\|\,\|x - Ay\| \\ &\leq \frac{M\,\|y\|}{\delta|z|} + \epsilon/2 < \epsilon,\end{aligned}$$

so that 2 holds.

3 implies 1. Suppose 3 holds for some $x \notin \overline{range(A)}$. By the Hahn–Banach theorem, there exists then $x^* \in X^*$ such that $x^*x = 1$ and $x^*(Ay) = 0$ for all $y \in D(A)$. Since $T(z)X \subset D(A)$ for all $z \in S_\alpha$,

$$\frac{d}{dz}x^*T(z)x = x^*AT(z)x = 0 \quad (z \in S_\alpha).$$

Hence

$$x^*T(z)x = x^*T(0)x = x^*x = 1 \quad (z \in S_\alpha).$$

This contradicts 3. □

Corollary 1.74. *Let $T(\cdot)$ be as in Theorem 1.69. Then the following statements are equivalent:*

1. *A has dense range.*
2. *$T(z) \to 0$ in the strong operator topology, as $z \to \infty$ within any proper subsector of S_θ.*
3. *There exists a sequence $\{z_k\}$ diverging to ∞ within some proper subsector of S_θ, such that $T(z_k) \to 0$ in the weak operator topology.*

Statement 2 (with $z \to \infty$ on the real axis) is usually referred to as *stability* of the semigroup. Statement 1 is equivalent to the spectral condition $0 \notin P\sigma(A) \cup R\sigma(A)$, where $P\sigma(A)$ and $R\sigma(A)$ denote the *point spectrum* and the *residual spectrum* of A, respectively.

E.2 Resolvent Iterates

Let A be any (generally unbounded) operator with nonempty resolvent set $\rho(A)$. It follows from the resolvent equation (cf. Theorem 1.11) and induction that

$$R(\lambda; A)^n = \frac{(-1)^{n-1}}{(n-1)!} R(\lambda; A)^{(n-1)} \quad (n \in \mathbb{N}). \tag{1}$$

In case A is the generator of a C_o-semigroup $T(\cdot)$, and a, M are such that $\|T(t)\| \le M\, e^{at}$ (cf. Theorem 1.1), it follows from the Laplace integral representation of the resolvent (cf. Theorem 1.15) and (1) that

$$\|[(\Re\lambda - a)\, R(\lambda; A)]^n\| \le M \quad (\Re\lambda > a;\, n \in \mathbb{N}). \tag{2}$$

(Conversely, if (2) holds, then A is necessarily closed since its resolvent set is nonempty (by (2)), and therefore, *in case A is densely defined*, it generates a C_o-semigroup $T(\cdot)$ such that $\|T(t)\| \le M\, e^{at}$, by Theorem 1.17.)

Motivated by the concept of an “abstract potential” (cf. paragraph following Theorem 1.15), we consider an *arbitrary* operator A whose resolvent exists and satisfies the growth condition

$$\|(\Re\lambda - a)\, R(\lambda; A)\| \le M \tag{3}$$

in a halfplane

$$\Pi_a := \{\lambda \in \mathbb{C};\, \Re\lambda > a\}.$$

The following result gives the asymptotic behavior of the resolvent iterates that replaces (2) in this more general situation.

Theorem 1.75. *Let A be a (generally unbounded) operator satisfying (3) in the halfplane Π_a. Then*

$$\|[(\Re\lambda - a)\, R(\lambda; A)]^n\| < Men \tag{4}$$

for all $n \in \mathbb{N}$ and $\lambda \in \Pi_a$.

Proof. Fix $n \in \mathbb{N}$, $n \geq 2$, and $\lambda \in \Pi_a$. For any $t \in (0,1)$, let Γ_t denote the positively oriented circle with center λ and radius $r = t(\Re\lambda - a)$. Clearly Γ_t and its interior are contained in Π_a. Since $R(\cdot; A)$ is analytic in Π_a, the Cauchy formula for the derivatives of analytic functions and (1) imply that

$$R(\lambda; A)^n = \frac{(-1)^{n-1}}{2\pi i} \int_{\Gamma_t} \frac{R(\mu; A)}{(\mu - \lambda)^n} \, d\mu. \tag{5}$$

For $\mu \in \Gamma_t$, we have

$$\Re\mu - a \geq \Re\lambda - r - a = (1-t)(\Re\lambda - a),$$

and therefore by (3)

$$\left\| \frac{R(\mu; A)}{(\mu - \lambda)^n} \right\| \leq \frac{M}{(\Re\mu - a) r^n} \leq \frac{M}{(\Re\lambda - a)^{n+1}(1-t)t^n}.$$

Since Γ_t has length $2\pi t(\Re\lambda - a)$, it follows from (5) that

$$\|[(\Re\lambda - a)\, R(\lambda; A)]^n\| \leq \frac{M}{(1-t)t^{n-1}}. \tag{6}$$

The left-hand side of (6) being independent of t, we may choose $t \in (0,1)$ such that the right-hand side of (6) is minimal. The absolute maximum of $(1-t)t^{n-1}$ in the interval $[0,1]$ is attained at $t = 1 - (1/n)$. This choice of t gives the estimate

$$\|[(\Re\lambda - a)\, R(\lambda; A)]^n\| \leq \frac{Mn}{[1 - (1/n)]^{n-1}},$$

and (4) follows then from the elementary inequality

$$[1 - (1/n)]^{n-1} > 1/e \quad (2 \leq n \in \mathbb{N}).$$ □

A "disk analog" of Theorem 1.75 is the following

Theorem 1.76. *Let A be a (generally unbounded) operator whose resolvent exists and satisfies the estimate*

$$\|(1 - |z|)\, R(z; A)\| \leq M \tag{7}$$

for all z in the (open) unit disk Ω. Then

$$\|[(1 - |z|)\, R(z; A)]^n\| < Men \tag{8}$$

for all $z \in \Omega$ and $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$, $n \geq 2$, $z \in \Omega$, and $t \in (0,1)$. The (positively oriented) circle Γ_t with center at z and radius $t(1-|z|)$ and its interior are contained in Ω. Hence (cf. (5))

$$R(z;A)^n = \frac{(-1)^{n-1}}{2\pi i} \int_{\Gamma_t} (\zeta - z)^{-n} R(\zeta;A)\, d\zeta.$$

For $\zeta \in \Gamma_t$, $\|R(\zeta;A)\| \leq M/(1-|\zeta|)$ by (7), and

$$1 - |\zeta| \geq 1 - [|z| + t(1-|z|)] = (1-t)(1-|z|).$$

Therefore

$$\|[(1-|z|)\, R(z;A)]^n\| \leq \frac{M}{(1-t)t^n}.$$

As in the preceding proof, we choose $t = 1 - (1/n)$ in order to minimize the right-hand side of the last inequality, hence getting the estimate

$$Mn/[1-(1/n)]^{n-1},$$

which is smaller than Men. □

In some special situations, the $O(n)$ estimate of Theorem 1.76 can be improved to a $O(1)$ estimate. For example, we have

Theorem 1.77. *Let A be a contraction on the Hilbert space X, whose resolvent exists and satisfies (7) in Ω. Then*

$$\|[(1-|z|)\, R(z;A)]^n\| = O(1) \quad (|z| \neq 1;\, n \in \mathbb{N}).$$

Proof. By a theorem of Gohberg and Krein (cf. [N1, p. 20]), the hypothesis of Theorem 1.77 implies that A is similar to a unitary operator U, that is, $A = Q^{-1}UQ$ for a suitable nonsingular operator $Q \in B(X)$. Let E be the resolution of the identity for U. Then for $|z| \neq 1$

$$R(z;A)^n = Q^{-1} \int_{\Gamma} \frac{E(d\zeta)}{(z-\zeta)^n}\, Q,$$

where Γ denotes the unit circle.

Since $|z - \zeta| \geq \big|1 - |z|\big|$ for $\zeta \in \Gamma$, it follows that

$$\|[(1-|z|)\, R(z;A)]^n\| \leq \|Q^{-1}\|\, \|Q\|$$

for all z, $|z| \neq 1$, and $n \in \mathbb{N}$. □

Theorem 1.76 is valid in the analogous version in which both the hypothesis and the conclusion relate to the domain $\Omega' := \{z \in \mathbb{C};\, |z| > 1\}$. The proof is identical. In particular, if A is a *bounded* operator on the Banach space X with spectrum in $\overline{\Omega}$ such that $\|(|z|-1)\, R(z;A)\| \leq M$ for $|z| > 1$, then $\|R(z;A)^n\| \leq Men/(|z|-1)^n$ for all $n \in \mathbb{N}$ and $z \in \Omega'$. Suppose f is a function analytic with modulus $\leq K$ in the disk $|z| < R$, for some $R > 1$. Since $\sigma(A)$ is contained in the closed unit disk, the operators $f^{(n)}(A)$ are well-defined by means of the Riesz–Dunford analytic operational calculus. For any $1 < r < R$, we have the formula (cf. [DS I–III, p. 591])

$$f^{(n)}(A) = \frac{n!}{2\pi i} \int_{|z|=r} f(z) R(z; A)^{n+1} dz. \tag{9}$$

Using the preceding estimate on the resolvent iterates, we obtain

$$\|f^{(n)}(A)\| \leq MKe(n+1)!r/(r-1)^{n+1}.$$

Since the left-hand side does not depend on r (as long as $1 < r < R$), and the right-hand side decreases with r, we obtain the best estimate of the above type by letting $r \to R-$, whence

$$\|f^{(n)}(A)\| \leq MKe \frac{(n+1)!R}{(R-1)^{n+1}} \quad (n \in \mathbb{N}). \tag{10}$$

These are Cauchy-type estimates for the operational calculus. Formally

Corollary 1.78. *Let A be a bounded operator in the Banach space X with spectrum in the closed unit disk, such that*

$$\|(|z|-1)R(z; A)\| \leq M \quad (|z| > 1).$$

Let f be analytic with $|f| \leq K$ in the disk $|z| < R$ (where $R > 1$). Then the "Cauchy estimates" (10) are valid.

Example 1.79. Let X be the Lebesgue space $L^p(0,1)$ for any p, $1 \leq p < \infty$, or the space of continuous functions $C([0,1])$. Let S be the multiplication operator $S : h(t) \to th(t)$, and let V be the classical Volterra operator $V : h(t) \to \int_0^t h(s)\,ds$. The spectrum of S is the interval $[0,1]$, and $[S,V] := SV - VS = V^2$. By Corollary 5.23 in [K4], $\sigma(S+V) = \sigma(S) = [0,1]$. Fix $c > 1$ and define

$$A = icI + S + V.$$

Then $\sigma(A) = ic + [0,1]$, and therefore $\Omega \subset \rho(A)$. By Corollary 5.21(c) in [K4] with $f(\zeta) = (z - ic - \zeta)^{-n}$, we have for $z \notin [0,1] + ic$

$$\begin{aligned} R(z; A)^n &= R(z - ic; S + V)^n = f(S+V) = f(S) + f'(S)V \\ &= R(z-ic; S)^n + nR(z-ic; S)^{n+1}V. \end{aligned} \tag{11}$$

Clearly

$$R(z - ic; S)^n : h(t) \to (z - ic - t)^{-n} h(t) \quad (t \in [0,1]). \tag{12}$$

For $z \in \Omega$ and $t \in [0,1]$, we have

$$|z - ic - t| \geq |t + ic| - |z| > 1 - |z|$$

(since $c > 1$), and also

$$|z - ic - t| \geq c - |z| > c - 1.$$

Therefore, by (11) and (12),

$$\|R(z; A)^n\| \leq (1 - |z|)^{-n} \left[1 + \frac{n\,\|V\|}{c - 1}\right]$$

for all $z \in \Omega$ and $n \in \mathbb{N}$. Thus A satisfies (7) with $M = 1 + \|V\|/(c - 1)$, and since $1 + n\,\|V\|/(c - 1) \leq Mn$, we have the estimates (8) of Theorem 1.76 with the factor e omitted. It is not known if the factor e in (8) is sharp (or if the $O(n)$ estimate of (8) is best possible in general under the hypothesis of the theorem).

E.3 Mean Stability

Returning to the asymptotic behavior of *general* C_o-semigroups, it turns out that "averaging" greatly simplifies the situation.

Notation 1.80. Given a set of conditions $\{1, 2, \dots\}$, $\mathcal{W}(1, 2, \dots)$ denotes the set of all functions $h(t, s)$ on $0 < s \leq t < \infty$ satisfying these conditions.

The following conditions will be mainly considered.

1. For all $t > 0$, $h(t, \cdot) \geq 0$ is monotonic on $(0, t]$.
2. $K := \sup_{t>0} \int_0^t h(t, s)\, ds < \infty$.
3. $\lim_{t\to\infty} h(t, t) = 0$.
4. There exists $\delta > 0$ such that $\lim_{t\to\infty} \int_0^\delta h(t, s)\, ds = 0$.
5. $\liminf_{t\to\infty} \int_0^t h(t, s)\, ds > 0$.

Conditions 1, 3, and 4 together imply that

$$\lim_{t\to\infty} h(t, \tau) = 0 \quad (\tau \geq \delta). \tag{1}$$

Indeed, if some $t > \tau \geq \delta$ is such that $h(t, \cdot)$ is nondecreasing (resp., nonincreasing), then $0 \leq h(t, \tau) \leq h(t, t)$ (resp., $0 \leq h(t, \tau) \leq h(t, \delta) \leq \delta^{-1} \int_0^\delta h(t, s)\, ds$), and therefore (1) follows from Conditions 3 and 4.

Such "weight functions" arise *for example* from monotonic one-variable functions $f : (0, 1] \to [0, \infty)$ such that

$$0 < \int_0^1 f(u)\, du := c < \infty, \tag{2}$$

by taking

$$h(t, s) := t^{-1} f(s/t) \quad (0 < s \leq t). \tag{3}$$

Such a weight function h satisfies Conditions 1–5, and even the stronger Condition $2'$ below instead of Conditions 2 and 5:

$2'$. For all $t > 0$, $\int_0^t h(t,s)\,ds = c$, where c is a positive finite constant.

For example, if we choose $f(u) = \beta u^{\beta-1}$ with $\beta > 0$, the induced weight functions are the classical "kernels" $h(t,s) = (\beta/t^\beta)s^{\beta-1}$ of fractional integration.

Notation 1.81. Let $F : [0,\infty) \to B(X)$ be a bounded strongly continuous function. If $h \in \mathcal{W}(1,2)$, set

$$(W_t^h F)x := \int_0^t h(t,s)\,F(s)x\,ds \quad (t > 0,\, x \in X).$$

If $M := \sup \|F(\cdot)\|$, the norm of the integrand above is $\le M\,\|x\|\,h(t,s)$, and it follows from Conditions 1 and 2 that the integral makes sense and defines bounded operators $W_t^h F$ with norm $\le MK$ for all $t > 0$.

Definition 1.82. *Let F be as in Notation 1.81. We say that F is stable (W^h-mean stable for some $h \in \mathcal{W}(1,2)$) if $\lim_{t\to\infty} F(t) = 0$ ($\lim_{t\to\infty} W_t^h F = 0$, respectively) in the strong operator topology (s.o.t.).*

Proposition 1.83. *Let $h \in \mathcal{W}(1,2,4)$. Then stability implies W^h-mean stability (for any F as in 1.81).*

Proof. Let F (as in Notation 1.81) be stable, and let $h \in \mathcal{W}(1,2,4)$. If δ is as in Condition 4 for h, $\delta < \tau < t$, and $x \in X$, we write

$$(W_t^h F)x = \left[\int_0^\delta + \int_\delta^\tau + \int_\tau^t\right] h(t,s)F(s)x\,ds.$$

Given $\epsilon > 0$, we may *fix* $\tau > \delta$ such that $K\,\|F(s)x\| < \epsilon$ for $s \ge \tau$ (by the stability of F). Then the third integral has norm $< \epsilon$ for all $t > \tau$. The first integral has norm $\le M\,\|x\| \int_0^\delta h(t,s)\,ds \to 0$ (as $t \to \infty$) by Condition 4. Finally, since $h(t,\cdot)$ is monotonic in $(0,t]$, the middle integral has norm

$$\le M\,\|x\|(\tau - \delta)\max[h(t,\tau),\, h(t,\delta)] \to 0$$

as $t \to \infty$, by (1). □

Theorem 1.84. *Let $T(\cdot)$ be a bounded C_o-semigroup, and let A be its generator. Then the following statements are equivalent:*

1. *A has dense range.*
2. *$T(\cdot)$ is W^h-mean stable for all weight functions $h \in \mathcal{W}(1,2,3,4)$.*
3. *For some weight function $h_0 \in \mathcal{W}(1,2,5)$ and some positive sequence $\{t_n\}$ diverging to ∞, $\lim_n W_{t_n}^{h_0} T(\cdot) = 0$ in the weak operator topology (w.o.t.).*

The theorem follows immediately from the following

Theorem 1.85. *Let $T(\cdot)$ be a bounded C_o-semigroup, and let A be its generator. Then the following statements are equivalent for any given vector $x \in X$:*

1. $x \in \overline{range\, A}$.
2. $\lim_{t\to\infty}[W_t^h T(\cdot)]x = 0$ *strongly for all* $h \in \mathcal{W}(1,2,3,4)$.
3. $\lim_n [W_{t_n}^{h_0} T(\cdot)]x = 0$ *weakly for some* $h_0 \in \mathcal{W}(1,2,5)$ *and some positive sequence* $\{t_n\}$ *diverging to* ∞.

Proof. Since 2 implies 3 trivially (because $\mathcal{W}(1,2,3,4,5) \neq \emptyset$), we need only to prove that 1 implies 2 and 3 implies 1.

1 implies 2. Since $\|W_t^h T(\cdot)\| \leq MK$ (where $M := \sup \|T(\cdot)\|$) for all $t > 0$ and $h \in \mathcal{W}(1,2)$, it follows from 1 that it suffices to prove the relation

$$\lim_{t\to\infty} [W_t^h T(\cdot)]\, Ay = 0 \tag{4}$$

for all $y \in D(A)$.

With δ as in Condition 4 on h and $t > \delta$, write

$$[W_t^h T(\cdot)]\, Ay = \left[\int_0^\delta + \int_\delta^t\right] h(t,s)T(s)Ay\, ds = J_1 + J_2.$$

We have

$$\|J_1\| \leq M\, \|Ay\| \int_0^\delta h(t,s)\, ds. \tag{5}$$

Since $T(\cdot)Ay = [T(\cdot)y]'$ for $y \in D(A)$, an integration by parts shows that

$$J_2 = h(t,t)T(t)y - h(t,\delta)T(\delta)y - \int_\delta^t T(s)y\, d_s h(t,s). \tag{6}$$

By the monotonicity assumption on h (cf. Condition 1), the Stieltjes integral in (6) exists and has norm $\leq M\, \|y\|\, |h(t,t) - h(t,\delta)|$. Therefore, by (5) and (6),

$$\|[W_t^h T(\cdot)]\, Ay\| \leq M\, \|Ay\| \int_0^\delta h(t,s)\, ds + 2M\, \|y\|\, [h(t,t) + h(t,\delta)] \to 0$$

as $t \to \infty$, by Conditions 3 and 4 on h and (1).

3 implies 1. Assume Statement 3 is valid for some h_0 and some sequence $\{t_n\}$ as described, and for some $x \notin \overline{range\, A}$. By the Hahn–Banach theorem, there exists $x^* \in X^*$ such that $x^*(Ay) = 0$ for all $y \in D(A)$ and $x^*x = 1$. Hence for all $y \in D(A)$ and $t > 0$,

$$\frac{d}{dt} x^* T(t)y = x^* AT(t)y = 0,$$

and therefore

$$x^*T(\cdot)y = \text{const.} = x^*T(0)y = x^*y.$$

Consequently, for all $y \in D(A)$ and $t > 0$,

$$x^*[W_t^{h_0}T(\cdot)]y = (x^*y)\int_0^t h_0(t,s)\,ds. \tag{7}$$

Since $D(A)$ is dense in X, it follows by continuity that (7) is valid for all $y \in X$ and $t > 0$. Taking $y = x$ and $t = t_n$ in (7), and letting $n \to \infty$, we conclude from Statement 3 that $\lim_n \int_0^{t_n} h_0(t_n, s)\,ds = 0$, contradicting Condition 5 on h_0. □

Remarks 1.86.

1. Statement 1 in Theorem 1.84 may be formulated as the spectral condition $0 \notin P\sigma(A) \cup R\sigma(A)$.
2. By Theorem 1.84, for any weight function $h \in \mathcal{W}(1,2,3,4,5)$, the concept of W^h-mean stability of a bounded C_o-semigroup (defined originally with respect to the strong operator topology) *is the same with respect to the s.o.t. and with respect to the w.o.t.*
3. The change of variables $u = t - s$ yields analogous results for "weight functions" $h(t,s)$ defined for $0 \le s < t < \infty$ (that is, undefined for $s = t$), satisfying the corresponding "shifted" conditions. This would apply for example to the usual fractional integrals of $T(\cdot)$.

The mean operators W^h are of "Cesaro type." We may also consider mean operators that generalize Abel or Gauss means. The weight functions $h : (0,\infty)^2 \to [0,\infty)$ are required to satisfy some of the following conditions:

(a) For each $t > 0$, $h(t,\cdot)$ is nonincreasing.
(b) $K := \sup_{t>0} \int_0^\infty h(t,s)\,ds < \infty$.
(c) There exists $\delta > 0$ such that $\lim_{t\to\infty} \int_0^\delta h(t,s)\,ds = 0$.
(d) $\liminf_{t\to\infty} \int_0^\infty h(t,s)\,ds > 0$.

The class of weight functions h satisfying Conditions (a),(b), etc. will be denoted $\mathcal{A}(a,b,\dots)$.

Let $h \in \mathcal{A}(a,c)$. If δ is as in Condition (c), then for all $v \ge \delta$

$$\lim_{t\to\infty} h(t,v) = 0 \tag{8}$$

because

$$0 \le h(t,v) \le h(t,\delta) \le \delta^{-1}\int_0^\delta h(t,s)\,ds \to 0$$

as $t \to \infty$.

As before, examples of weight functions $h \in \mathcal{A}(a,b,c,d)$ are induced by one-variable nonincreasing functions $f : [0,\infty) \to [0,\infty)$, by taking $h(t,s) := (1/t)f(s/t)$. Thus, if $f(u) = e^{-u}$ or $f(u) = (2/\sqrt{\pi})\, e^{-u^2}$, we get the classical Abel and Gauss summability kernels, $h(t,s) = (1/t)e^{-s/t}$ and $h(t,s) = (2/t\sqrt{\pi})\, e^{-(s/t)^2}$, respectively. Weight functions induced in this manner satisfy even the Condition (b′) below, which is stronger than Conditions (b) and (d) together:

(b′) For all $t > 0$, $\int_0^\infty h(t,s)\, ds = c$, where $c > 0$ denotes a constant.

Let $F : [0,\infty) \to B(X)$ be a bounded strongly continuous function. If $h \in \mathcal{A}(a,b)$, the A^h-means of F are the well-defined bounded operators given by

$$(A_t^h F)x := \int_0^\infty h(t,s)F(s)x\, ds \quad (x \in X;\, t > 0).$$

By Conditions (a) and (b), $\|A_t^h F\| \le MK$ for all $t > 0$, where $M := \sup \|F\|$. We say that F is *A^h-mean stable* if $\lim_{t\to\infty} A_t^h F = 0$ in the strong operator topology. The analogues of Theorems 1.84 and 1.85 are stated below.

Theorem 1.87. *Let $T(\cdot)$ be a bounded C_o-semigroup, and let A be its generator. Then the following statements are equivalent:*

1. *range A is dense in X.*
2. *$T(\cdot)$ is A^h-mean stable for all $h \in \mathcal{A}(a,b,c)$.*
3. *For some weight function $h_0 \in \mathcal{A}(a,b,d)$ and for some positive sequence $\{t_n\}$ diverging to ∞, $\lim_n A_{t_n}^{h_0} T(\cdot) = 0$ in the w.o.t.*

Theorem 1.88. *Let $T(\cdot)$ be a bounded C_o-semigroup, and let A be its generator. Then the following statements are equivalent for a given vector $x \in X$:*

1. *$x \in \overline{\text{range}\, A}$.*
2. $\lim_{t\to\infty} [A_t^h T(\cdot)]x = 0$ *strongly for all $h \in \mathcal{A}(a,b,c)$.*
3. $\lim_n [A_{t_n}^{h_0} T(\cdot)]x = 0$ *weakly for some weight function $h_0 \in \mathcal{A}(a,b,d)$ and for some positive sequence $\{t_n\}$ diverging to ∞.*

Proof. Since Theorem 1.88 implies Theorem 1.87, and Statement 3 implies Statement 1 in Theorem 1.88 in the same manner as in Theorem 1.85 (with the means A^h replacing the means W^h), it suffices to prove that 1 implies 2 in Theorem 1.88. As before, this will be achieved by showing that

$$\lim_{t\to\infty} \|[A_t^h T(\cdot)]\, Ay\| = 0 \quad (y \in D(A)) \tag{9}$$

for all $h \in \mathcal{A}(a,b,c)$.

Given $h \in \mathcal{A}(a,b,c)$, $y \in D(A)$, and δ as in Condition (c), we write

$$[A_t^h T(\cdot)]\, Ay = \int_0^\delta h(t,s)T(s)Ay\, ds + \lim_{v\to\infty} \int_\delta^v h(t,s) \frac{d}{ds}[T(s)y]\, ds. \tag{10}$$

The first summand on the right-hand side has norm

$$\leq M \, \|Ay\| \int_0^\delta h(t,s)\, ds.$$

For $v > \delta$, the integral over $[\delta, v]$ in (10) can be written as

$$h(t,v)T(v)y - h(t,\delta)T(\delta)y - \int_\delta^v T(s)y \, d_s h(t,s). \tag{11}$$

Since $h(t,\cdot)$ is nonincreasing, the norm of (11) is

$$\begin{aligned} &\leq M \, \|y\| [h(t,v) + h(t\delta) + (h(t,\delta) - h(t,v))|] \\ &\leq 2M \, \|y\| \, h(t,\delta). \end{aligned}$$

We then conclude from (10) and (11) that

$$\|[A_t^h T(\cdot)] \, Ay\| \leq M \, \|Ay\| \int_0^\delta h(t,s)\, ds + 2M \, \|y\| \, h(t,\delta) \to 0$$

as $t \to \infty$, by Condition (c) and (8). □

Let $T(\cdot)$ and $S(\cdot)$ be bounded C_o-semigroups on the Banach space X, and let A and $-B$ be their respective generators. We consider the operator Δ on $B(X)$ defined by

$$\Delta V := AV - VB$$

with the domain

$$D(\Delta) := \{V \in B(X);\, V\, D(B) \subset D(A), \quad \Delta V \in B(D(B), X)\}. \tag{12}$$

Since $D(B)$ is dense in X, it follows from (12) that if $V \in D(\Delta)$, then ΔV extends uniquely as an operator in $B(X)$ (also denoted by ΔV).

Set

$$G(t)V := T(t)VS(t) \quad (t \geq 0,\, V \in B(X)). \tag{13}$$

Clearly, $G(\cdot)$ is a semigroup of operators on $B(X)$ such that $G(\cdot)V$ is continuous in the s.o.t. for each $V \in B(X)$. $G(\cdot)$ is called the *tensor product* of the given semigroups (cf. [Fre]). A minor modification of the preceding proofs yields the following result.

Theorem 1.89. *Let $T(\cdot)$ and $S(\cdot)$ be bounded C_o-semigroups, and let A and $-B$ be their respective generators. Let Z belong to the $B(X)$-closure of range Δ. Then $T(\cdot)ZS(\cdot)$ is W^h-mean stable (A^h-mean stable) for all $h \in \mathcal{W}(1,2,3,4)$ ($h \in \mathcal{A}(a,b,c)$, respectively).*

Proof. Since $\|W^h[T(\cdot)ZS(\cdot)]\| \le M_1M_2K\,\|Z\|$ (where M_1 and M_2 are bounds for $\|T(\cdot)\|$ and $\|S(\cdot)\|$, respectively), and similarly for the A^h means, it suffices to consider Z in the range of Δ, say $Z = \Delta V$ with $V \in D(\Delta)$. By Proposition 5 in [Fre], $G(\cdot)V$ is then differentiable in the s.o.t. on $[0,\infty)$, and its s.o.t.-derivative is equal to $G(\cdot)Z$ $(= T(\cdot)ZS(\cdot))$. The integration by parts argument in the proofs of Theorems 1.85 and 1.88 yields the result for the W^h-means and the A^h-means, respectively. □

Corollary 1.90. *Let $S(\cdot)$ and $T(\cdot)$ be bounded C_o-semigroups, and let $-B$ and $B+C$ be their respective generators, where $C \in B(X)$. Then $T(\cdot)CS(\cdot)$ is W^h-mean (A^h-mean) stable for all $h \in \mathcal{W}(1,2,3,4)$ ($h \in \mathcal{A}(a,b,c)$, respectively).*

Proof. Since $C \in B(X)$, it follows that $I \in D(\Delta)$, and we may take $Z := \Delta I = (B+C) - B = C$ in Theorem 1.89. □

Discrete analogs of Theorems 1.85 and 1.88 can be formulated for *power bounded operators*, that is, operators T such that

$$M := \sup_{n\in\mathbb{N}} \|T^n\| < \infty. \tag{14}$$

Notation 1.91. $\mathcal{T}(1,2,\dots)$ denotes the set of all infinite triangular matrices $W = (w_{nk})_{1\le k\le n<\infty}$ with the following properties:

1. For each $n \in \mathbb{N}$, $w_{nk} \ge 0$ is monotonic with respect to k, $k = 1,\dots,n$.
2. $K := \sup_{n\in\mathbb{N}} c_n(W) < \infty$, where $c_n(W) := \sum_{k=1}^n w_{nk}$.
3. $\lim_n w_{n1} = \lim_n w_{nn} = 0$.
4. $\liminf_n c_n(W) > 0$.

For example, if $f : (0,1] \to [0,\infty)$ is a monotonic function such that $tf(t) \to 0$ as $t \to 0+$ and $\int_0^1 f(t)\,dt := c \ne 0$, then the matrix with the entries $w_{nk} = (1/n)f(k/n)$ $(1 \le k \le n < \infty)$ satisfies Conditions 1–4, and even the following Condition 2′ (stronger than 2 and 4 together):

2′. $\exists \lim_n c_n(W) := c \ne 0$.

The W-means of the power sequence $\{T^k\}$ with respect to the "weight matrix" $W \in \mathcal{T}(\cdots)$ are defined by

$$W_n(T) = \sum_{k=1}^n w_{nk}T^k \quad (n \in \mathbb{N}).$$

The operator T is *W-mean stable* if $W_n(T) \to 0$ in the s.o.t. The discrete analogs of Theorems 1.84 and 1.85 are given below.

Theorem 1.92. *Let $T \in B(X)$ be power bounded. Then the following statements are equivalent for a vector $x \in X$:*

(a) $x \in \overline{range(I-T)}$.
(b) *For all* $W \in \mathcal{T}(1,2,3)$, $\lim_n W_n(T)x = 0$ *strongly.*
(c) *For some* $\tilde{W} \in \mathcal{T}(1,2,4)$ *and some subsequence* $\{n_k\}$, $\lim_k \tilde{W}_{n_k}(T)x = 0$ *weakly.*

Theorem 1.93. *Let* $T \in B(X)$ *be power bounded. Then the following statements are equivalent:*

(a) $I - T$ *has dense range.*
(b) T *is* W*-mean stable for all* $W \in \mathcal{T}(1,2,3)$.
(c) *For some* $\tilde{W} \in \mathcal{T}(1,2,4)$ *and some subsequence* $\{n_k\}$, $\lim_k \tilde{W}_{n_k}(T) = 0$ *in the w.o.t.*

Since Theorem 1.93 clearly follows from Theorem 1.92, the latter is proved below.

Proof. Since (b) implies (c) trivially (because $\mathcal{T}(1,2,3,4) \neq \emptyset$), we need to prove that (a) implies (b) and (c) implies (a).

(a) *implies* (b). Let $W \in \mathcal{T}(1,2,3)$ and $M := \sup_k \|T^k\|$. Since $\|W_n(T)\| \leq MK$ for all $n \in \mathbb{N}$ (by Condition 2), it suffices to prove that $\lim_n [W_n(T)](I-T)y = 0$ for all $y \in X$.

For $n \geq 2$, we have by Abel's summation formula

$$W_n(T)(I-T) = \sum_{k=1}^{n} w_{nk}(T^k - T^{k+1})$$

$$= w_{n1}T - w_{nn}T^{n+1} + \sum_{k+2}^{n}(w_{nk} - w_{n,k-1})T^k.$$

By the monotonicity assumption in Condition 1,

$$\left\|\sum_{k=2}^{n}(\cdots)\right\| \leq M \sum_{k=2}^{n} |w_{nk} - w_{n,k-1}|$$

$$= M \left|\sum_{k=2}^{n}(w_{nk} - w_{n,k-1})\right| = M\,|w_{nn} - w_{n1}|.$$

Hence

$$\|W_n(T)(I-T)\| \leq 2M\,(w_{n1} + w_{nn}) \to 0$$

as $n \to \infty$, by Condition 3.

(c) *implies* (a). Suppose (c) holds for the given vector x, but $x \notin \overline{range\,(I-T)}$. By the Hahn–Banach theorem, there exists $x^* \in X^*$ such that $x^*x = 1$ and $x^*(I-T)y = 0$ for all $y \in X$. Then $T^*x^* = x^*$, and therefore, for any (triangular) matrix W,

$$[W_n(T)]^*x^* = c_n(W)x^* \quad (n \in \mathbb{N}).$$

Consequently

$$x^*W_n(T)x = ([W_n(T)]^*x^*)x = c_n(W)x^*x = c_n(W).$$

Taking $W = \tilde{W}$ and $n = n_k$ as in (c), we get $c_{n_k}(\tilde{W}) \to 0$ as $k \to \infty$, contradicting Condition 4. □

For discrete analogues of Theorems 1.87 and 1.88, we consider infinite *square* matrices $A = (a_{nk})_{n,k\in\mathbb{N}}$ with some of the following properties:

1. For each $n \in \mathbb{N}$, $0 \le a_{nk}$ is monotonic with respect to $k \in \mathbb{N}$, and $\lim_k a_{nk} = 0$.
2. $\sup_n c_n(A) = K < \infty$, where $c_n(A) := \sum_{k=1}^{\infty} a_{nk}$.
3. $\lim_n a_{n1} = 0$.
4. $\liminf_n c_n(A) > 0$.

For example, if $f : (0,\infty) \to [0,\infty)$ is a nonincreasing function such that $f(\infty) = 0, \quad tf(t) \to 0$ as $t \to 0+$, and $\int_0^\infty f(t)\,dt = c \neq 0$, the matrix A with entries $a_{nk} = (1/n)f(k/n)$ satisfies Conditions 1–4, and even the following Condition $2'$ (stronger than Conditions 2 and 4 together):

$2'$. $\exists \lim_n c_n(A) = c \neq 0$.

The classical Abel and Gauss matrices are induced as above from the functions $f(u) = e^{-u}$ and $f(u) = (2/\sqrt{\pi})e^{-u^2}$, respectively.

In the present context, $\mathcal{M}(1, 2, \ldots)$ will denote the set of all infinite square matrices $A = (a_{nk})$ satisfying Conditions $1, 2, \ldots$. If $T \in B(X)$ is power bounded and $A \in \mathcal{M}(1, 2)$, the *A-means* $A_n(T)$ are the well-defined bounded operators

$$A_n(T) := \sum_{k=1}^{\infty} a_{nk}T^k \quad (n \in \mathbb{N}),$$

where the series converges in operator-norm, and $\|A_n(T)\| \le MK$ (with $M := \sup_k \|T^k\|$). We say that T is *A-mean stable* if $A_n(T) \to 0$ in the s.o.t.

Theorem 1.94. *Let $T \in B(X)$ be power bounded. Then the following statements are equivalent:*

(a) *$I - T$ has dense range.*
(b) *T is A-mean stable for all $A \in \mathcal{M}(1, 2, 3)$.*
(c) *For some $\tilde{A} \in \mathcal{M}(1, 2, 4)$ and some subsequence $\{n_k\}$, $\lim_k \tilde{A}_{n_k}(T) = 0$ in the w.o.t.*

Theorem 1.95. *Let $T \in B(X)$ be power bounded. Then the following statements are equivalent for a given vector $x \in X$:*

(a) *$x \in \overline{range\,(I - T)}$.*
(b) *$strong - \lim_n [A_n(T)]x = 0$ for all $A \in \mathcal{M}(1, 2, 3)$.*

(c) *For some* $\tilde{A} \in \mathcal{M}(1,2,4)$ *and some subsequence* $\{n_k\}$, $\lim_k [\tilde{A}_{n_k}(T)]x = 0$ *weakly.*

Clearly, Theorem 1.94 follows from Theorem 1.95, whose proof is given below.

Proof. (a) *implies* (b). Since $\|A_n(T)\| \le MK$ for all $n \in \mathbb{N}$, it suffices to prove that $\lim_n A_n(T)(I-T) = 0$. For $n \ge 2$, Abel's summation formula and Conditions 1 and 3 give

$$\begin{aligned}\|A_n(T)(I-T)\| &= \left\| a_{n1}T + \sum_{k=2}^{\infty}(a_{nk} - a_{n,k-1})T^k \right\| \\ &\le M\,a_{n1} + M\sum_{k=2}^{\infty}|a_{nk} - a_{n,k-1}| \\ &= M\,a_{n1} + M\left|\sum_{k=2}^{\infty}(a_{nk} - a_{n,k-1})\right| = 2M\,a_{n1} \to 0\end{aligned}$$

as $n \to \infty$.

(c) *implies* (a). The argument in the proof of Theorem 1.92 is valid in the present case. □

E.4 The Asymptotic Space

We turn now to the *stability* problem for general C_o-semigroups (cf. Corollary 1.74 for holomorphic semigroups and Definition 1.82). By the Uniform Boundedness Theorem, the stability of the C_o-semigroup $T(\cdot)$ implies the boundedness of $\|T(\cdot)\|$; it suffices therefore to consider the stability problem for *bounded* semigroups.

We begin with the special case of a *contraction* C_o-semigroup $T(\cdot)$, that is, $\|T(\cdot)\| \le 1$. (The general case will be easily reduced to this special case.)

We first construct the so-called *asymptotic space* of a given C_o-semigroup of contractions $T(\cdot)$.

If $t > s \ge 0$, then for all $x \in X$,

$$\|T(t)x\| = \|T(t-s)T(s)x\| \le \|T(t-s)\|\,\|T(s)x\| \le \|T(s)x\|,$$

that is, $\|T(\cdot)x\|$ is *nonincreasing*, and therefore

$$\exists \lim_{t\to\infty} \|T(t)x\| := |x|_a. \tag{1}$$

Clearly $|\cdot|_a$ is a seminorm on X, majorized by the given norm; we call it the *asymptotic seminorm* (with respect to $T(\cdot)$). The kernel K of the seminorm is a (closed) subspace of X. Let $\pi : X \to X/K$ be the quotient map, and define

$$\|\pi x\|_a := |x|_a \quad (x \in X). \tag{2}$$

Then $\|\cdot\|_a$ is a well-defined (i.e., independent on coset representatives) *norm* on X/K. We let Y denote the *completion* of the normed space $(X/K, \|\cdot\|_a)$. The Banach space Y will be called *the asymptotic space* of $T(\cdot)$; its norm, also denoted by $\|\cdot\|_a$, is the *asymptotic norm* (with respect to $T(\cdot)$). By definition, $Y \neq \{0\}$ iff the semigroup $T(\cdot)$ is *not* stable. *This will be our standing hypothesis* in the following leasurely discussion.

The semigroup $T(\cdot)$ induces a function $U(\cdot)$ on Y in the usual way. First, "define" $U(\cdot)$ on X/K by

$$U(t)(\pi x) := \pi T(t)x \quad (x \in X;\, t \geq 0). \tag{3}$$

Then by (1)–(3), for each $t \geq 0$ and $x \in X$,

$$\begin{aligned}\|U(t)(\pi x)\|_a &= \|\pi T(t)x\|_a = |T(t)x|_a \\ &= \lim_{s\to\infty} \|T(s)T(t)x\| = \lim_{u\to\infty} \|T(u)x\| = |x|_a = \|\pi x\|_a.\end{aligned}$$

Hence $U(t)$ is a well-defined isometry on X/K, and extends therefore (uniquely) as an *isometry* on the completion Y (same notation). Clearly, $U(\cdot)$ is a semigroup of operators on Y. Given $x \in X$ and $\epsilon > 0$, let $\delta > 0$ be such that $\|T(t)x - x\| < \epsilon$ for $0 \leq t < \delta$. Then for such t,

$$\begin{aligned}\|U(t)\,\pi x - \pi x\|_a &= \|\pi[T(t)x - x]\|_a = |T(t)x - x|_a \\ &\leq \|T(t)x - x\| < \epsilon,\end{aligned}$$

and the C_o-property for $U(\cdot)$ on Y follows. Denote by A, B the generators of the C_o-semigroups $T(\cdot)$ and $U(\cdot)$ on X and Y (respectively).

Lemma 1.96. *Let $T(\cdot)$ be a C_o-semigroup of contractions on the Banach space X, and let $U(\cdot)$ be the induced C_o-semigroup of isometries on the asymptotic space Y of $T(\cdot)$. Let A, B be the generators of $T(\cdot)$, $U(\cdot)$, respectively. Then $\pi D(A)$ is a dense subspace of $[D(B)]$ (where $[D(B)]$ denotes the Banach space $D(B)$ with the B-graph norm $\|y\|_B := \|y\|_a + \|By\|_a$), and $B(\pi x) = \pi(Ax)$ for all $x \in D(A)$.*

Proof. Since $\|T(\cdot)\| \leq 1 = \|U(\cdot)\|$, the (open) right halfplane $\mathbb{C}^+$ is contained in the resolvent sets of both A and B. For any $\lambda \in \mathbb{C}^+$,

$$R(\lambda; B)\,\pi x = \pi R(\lambda; A)x \quad (x \in X), \tag{4}$$

where $R(\lambda; A)$ and $R(\lambda; B)$ are the resolvent operators for A and B on the respective spaces X and Y. Indeed, by the Laplace integral representation of the resolvents (in the respective spaces) and the fact that $\pi : X \to (X/K, \|\cdot\|_a)$ is norm-decreasing:

$$\begin{aligned}R(\lambda; B)\,\pi x &= \int_0^\infty e^{-\lambda t} U(t)\,\pi x\,dt \\ &= \int_0^\infty e^{-\lambda t} \pi T(t)x\,dt = \pi \int_0^\infty e^{-\lambda t} T(t)x\,dt = \pi R(\lambda; A)x.\end{aligned}$$

(The validity of (4) is extended to all $\lambda \in \rho(A)$ in the next lemma.) It follows from (4) that

$$\pi D(A) \subset D(B). \tag{5}$$

Indeed, for any fixed $\lambda \in \mathbb{C}^+$, we have by (4)

$$\pi D(A) = \pi R(\lambda; A)X = R(\lambda; B)\,\pi X \subset R(\lambda; B)Y = D(B).$$

If $x \in D(A)$ (so that $\pi x \in D(B)$, by (5)), we have

$$B\,\pi x = \pi A x. \tag{6}$$

To see this, write $x = R(\lambda; A)z$ for any (fixed) $\lambda \in \mathbb{C}^+$ and the (unique) associated $z \in X$. Then by (4)

$$\begin{aligned} B\,\pi x &= B\,\pi R(\lambda; A)z = B\,R(\lambda; B)\,\pi z = \lambda R(\lambda; B)\,\pi z - \pi z \\ &= \pi\,[\lambda R(\lambda; A)z - z] = \pi A R(\lambda; A)z = \pi A x. \end{aligned}$$

(Of course, (5)–(6) could also be verified directly from the definition of the generators.)

We now verify that in addition to the inclusion (5), $\pi D(A)$ is *dense* in the Banach space $[D(B)]$, that is, the space $D(B)$ with the B-graph norm

$$y \in D(B) \subset Y \to \|y\|_B := \|y\|_a + \|By\|_a. \tag{7}$$

Consider an arbitrary $y \in D(B)$. Fix $\lambda \in \mathbb{C}^+$, and write $y = R(\lambda; B)z$ for a unique associated $z \in Y$. By definition of Y, there exists a sequence $\{v_n\} \subset X$ such that $\pi v_n \to z$ in $\|\cdot\|_a$-norm. Define $x_n := R(\lambda; A)v_n$. Then $x_n \in D(A)$, $\pi x_n \in D(B)$ (by (5)), and by (4) and the fact that $R(\lambda; B)$ is a bounded operator on $(Y, \|\cdot\|_a)$, we have

$$\pi x_n = \pi R(\lambda; A)v_n = R(\lambda; B)\,\pi v_n \to R(\lambda; B)z = y \tag{8}$$

in the $\|\cdot\|_a$-norm. Moreover, by (8)

$$\begin{aligned} B\,\pi x_n &= BR(\lambda; B)\,\pi v_n = \lambda R(\lambda; B)\,\pi v_n - \pi v_n \\ &\to \lambda R(\lambda; B)z - z = BR(\lambda; B)z = By \end{aligned}$$

in the $\|\cdot\|_a$-norm. Together with (8), this proves that $\pi x_n \to y$ in the B-graph norm, and we conclude that $\pi D(A)$ is dense in $[D(B)]$. □

Lemma 1.97 (Assumptions and notation as in Lemma 1.96). *$\rho(A) \subset \rho(B)$, and $R(\lambda; B)(\pi x) = \pi R(\lambda; A)x$ for all $\lambda \in \rho(A)$ and $x \in X$.*

Proof. Let $\lambda \in \rho(A)$. Define the operator $R(\lambda)$ on X/K by $R(\lambda)\,\pi x := \pi R(\lambda; A)x$. Then by (1) and (2),

$$\|R(\lambda)\,\pi x\|_a = |R(\lambda;A)x|_a := \lim_{t\to\infty} \|T(t)R(\lambda;A)x\|$$
$$= \lim_t \|R(\lambda;A)\,T(t)x\| \le \|R(\lambda;A)\|\,|x|_a = \|R(\lambda;A)\|\,\|\pi x\|_a,$$

i.e., $R(\lambda) \in B(X/K)$, with operator norm $\le \|R(\lambda;A)\|$. Therefore $R(\lambda)$ extends uniquely as a bounded operator on Y (same notation), with operator norm $\le \|R(\lambda;A)\|$. For all $x \in X$, by Lemma 1.96,

$$R(\lambda)\,\pi x := \pi R(\lambda;A)x \in \pi D(A) \subset D(B), \tag{9}$$

and

$$(\lambda I - B)\,R(\lambda)\,\pi x = (\lambda I - B)\pi R(\lambda;A)x = \pi(\lambda I - A)R(\lambda;A)x = \pi x, \tag{10}$$

where I denotes the identity operator on the appropriate space.

For $x \in D(A)$, $\pi x \in D(B)$ (by Lemma 1.96), and

$$R(\lambda)(\lambda I - B)\pi x = R(\lambda)\,\pi(\lambda I - A)x = \pi R(\lambda;A)(\lambda I - A)x = \pi x. \tag{11}$$

The injection operator and $R(\lambda)(\lambda I - B)$ are both bounded operators from $\pi D(A) \subset [D(B)]$ to Y, and coincide by (11) on the dense subspace $\pi D(A)$ of $[D(B)]$ (cf. Lemma 1.96). Therefore $R(\lambda)(\lambda I - B)y = y$ for all $y \in D(B)$.

On the other hand, for any $y \in Y$, there exist $x_n \in X$ such that $\pi x_n \to y$ in $\|\cdot\|_a$-norm. Hence $R(\lambda)\,\pi x_n \in D(B)$ (by (9)), $R(\lambda)\,\pi x_n \to R(\lambda)y$ in Y (since $R(\lambda) \in B(Y)$), and by (10), $(\lambda I - B)\,R(\lambda)\,\pi x_n = \pi x_n \to y$ in Y. Since $\lambda I - B$ is a closed operator on Y (with domain $D(B)$), it follows that $R(\lambda)y \in D(B)$ and $(\lambda I - B)R(\lambda)y = y$. In conclusion, we proved that the bounded operator $R(\lambda)$ on Y satisfies the relations

$$R(\lambda)(\lambda I - B)y = y \quad (y \in D(B))$$

and for all $y \in Y$,

$$R(\lambda)y \in D(B); \quad (\lambda I - B)R(\lambda)y = y,$$

that is, $\lambda \in \rho(B)$. Hence

$$\rho(A) \subset \rho(B) \tag{12}$$

(equivalently,

$$\sigma(B) \subset \sigma(A)), \tag{13}$$

and

$$R(\lambda;B) = R(\lambda) \quad (\lambda \in \rho(A)). \tag{14}$$

Restricting (14) to πX, this means that (4) is valid for all $\lambda \in \rho(A)$ ($\subset \rho(B)$) (and not merely for $\lambda \in \mathbb{C}^+$). □

E.5 Semigroups of Isometries

We shall need some general facts about C_o-semigroups of isometries.

Lemma 1.98. *Let $U(\cdot)$ be any C_o-semigroup of isometries on the Banach space Y, and let B be its generator. Then for all $\lambda \in \mathbb{C}$ and $y \in D(B)$,*

$$\|(\lambda I - B)y\| \geq |\Re\lambda| \, \|y\|.$$

Proof. For any $\lambda \in \mathbb{C}$, the C_o-semigroup $s \to e^{-\lambda s}U(s)$ has the generator $B - \lambda I$. Therefore

$$\frac{d}{ds}[e^{-\lambda s}U(s)y] = e^{-\lambda s}U(s)\,[B - \lambda I]y \quad (y \in D(B)).$$

Integrating over the interval $[0, t]$ $(t > 0)$, we obtain

$$e^{-\lambda t}U(t)y = y + \int_0^t e^{-\lambda s}U(s)[B - \lambda I]y \, ds. \tag{1}$$

Taking Y-norms, we get

$$\begin{aligned} e^{-\Re\lambda t}\|y\| &\leq \|y\| + \int_0^t e^{-\Re\lambda s}\|U(s)\,[B - \lambda I]y\| \, ds \\ &= \|y\| + \int_0^t e^{-\Re\lambda s}\, ds \, \|(B - \lambda I)y\|, \end{aligned}$$

that is, for $\Re\lambda \neq 0$,

$$[e^{-\Re\lambda t} - 1]\|y\| \leq \frac{e^{-\Re\lambda t} - 1}{-\Re\lambda}\|(B - \lambda I)y\|.$$

For $\Re\lambda < 0$, we divide the last inequality by the positive factor $e^{-\Re\lambda t} - 1$ to obtain the relation

$$\|(\lambda I - B)y\| \geq |\Re\lambda| \, \|y\| \quad (y \in D(B)). \tag{2}$$

For $\Re\lambda > 0$, (2) is valid in general for the generator B of *any contraction semigroup $S(\cdot)$ on any Banach space Y*. Indeed, such λ are in $\rho(B)$ and by the Laplace integral representation of the resolvent,

$$\begin{aligned} \|R(\lambda; B)z\| &= \left\| \int_0^\infty e^{-\lambda t}S(t)z \, dt \right\| \\ &\leq \int_0^\infty e^{-\Re\lambda t} \, dt \, \|z\| = (1/\Re\lambda) \, \|z\| \end{aligned}$$

for all $z \in Y$. If $y \in D(B)$, take $z = (\lambda I - B)y$ in the last inequality to get (2). □

Lemma 1.99. *Let B be any closed operator on a Banach space Y, which satisfies (2). Then for any $\lambda \in \mathbb{C}$ with $\Re\lambda \neq 0$, $\lambda I - B$ is injective and has closed range.*

Proof. Fix λ with $\Re\lambda \neq 0$. By (2), $\lambda I - B$ is trivially one-to-one. Suppose $z_n \in (\lambda I - B)Y$ and $z_n \to z$. Write $z_n = (\lambda I - B)y_n$ with $y_n \in D(B)$. By (2), for $n, m \to \infty$,

$$|\Re\lambda|\,\|y_n - y_m\| \leq \|(\lambda I - B)(y_n - y_m)\| = \|z_n - z_m\| \to 0,$$

hence $\exists \lim_n y_n := y$. Now $y_n \in D(B) = D(\lambda I - B)$, $y_n \to y$, and $(\lambda I - B)y_n = z_n \to z$. Since $\lambda I - B$ is closed, it follows that $y \in D(B)$ and $z = (\lambda I - B)y \in (\lambda I - B)Y$. We thus proved that $(\lambda I - B)$ *has a closed range.* □

Remark. Under the last lemma's assumptions, the operator

$$\lambda I - B : [D(B)] \to \operatorname{ran}(\lambda I - B)$$

is continuous and bijective between *Banach spaces*. By the Banach theorem on inverses, the inverse

$$(\lambda I - B)^{-1} : \operatorname{ran}(\lambda I - B) \to [D(B)]$$

is bounded, i.e.,

$$\|(\lambda I - B)^{-1} z\|_B \leq K\,\|z\| \quad (z \in \operatorname{ran}(\lambda I - B))$$

for a suitable constant $K > 0$. Equivalently, one has the relation

$$\|(\lambda I - B)y\| \geq (1/K)\|y\|_B \quad (y \in D(B)), \tag{3}$$

which is stronger than (2), since the graph norm is larger than the given norm.

Lemma 1.100. *Let $U(\cdot)$ be any C_o-semigroup of isometries on a nontrivial Banach space Y, and let B be its generator. Then either*

(a) *$\sigma(B)$ is the entire closed left halfplane; or*
(b) *$\sigma(B)$ is a nonempty subset of the imaginary axis. In this case, $U(\cdot)$ extends as a C_o-group of isometries on Y.*

Proof. Let λ (with $\Re\lambda < 0$) be a boundary point of $\sigma(B)$. Let then $\lambda_n \in \rho(B)$ converge to λ. By Theorem 1.11,

$$\|R(\lambda_n; B)\| \geq 1/[d(\lambda_n, \sigma(B))] \geq 1/|\lambda_n - \lambda| \to \infty.$$

Thus $\sup_n \|R(\lambda_n; B)\| = \infty$, and therefore, by the Uniform Boundedness Theorem, there exists $y \in Y$ such that $\sup_n \|R(\lambda_n; B)y\| = \infty$. Consequently, there exists a subsequence (say $\{\lambda_n\}$ itself, without loss of generality) such that $\|R(\lambda_n; B)y\| \to \infty$. Define

$$y_n := \|R(\lambda_n; B)y\|^{-1} R(\lambda_n; B)y.$$

Then $y_n \in D(B)$, $\|y_n\| = 1$, and

$$\begin{aligned}\|(\lambda I - B)y_n\| &\leq \|(\lambda_n I - B)y_n\| + \|(\lambda - \lambda_n)y_n\| \\ &= \|R(\lambda_n; B)\|^{-1}\|y\| + |\lambda - \lambda_n| \to 0.\end{aligned}$$

However, by Lemma 1.98,

$$\|(\lambda I - B)y_n\| \geq |\Re\lambda| > 0,$$

contradiction! This shows that *there are no boundary points of* $\sigma(B)$ *in the open left halfplane*. Since $\sigma(B)$ is contained in the closed left halfplane (because B generates the contraction C_o-semigroup $U(\cdot)$), it follows that either Case (a) holds, or else $\sigma(B) \subset i\mathbb{R}$.

In the latter case, the right halfplane is contained in $\rho(-B)$, and for all $\lambda > 0$, we have by Lemma 1.98

$$\|\lambda R(\lambda; -B)\| = \|\lambda R(-\lambda; B)\| \leq 1.$$

By Corollary 1.18, $-B$ generates a C_o-semigroup of contractions, and therefore $U(\cdot)$ extends to a C_o*-group* of contractions (cf. paragraph preceding Theorem 1.39), hence to a C_o*-group of isometries*.

If $\sigma(B) = \emptyset$, $R(\cdot; B)$ is an entire function (with values in the Banach space $B(Y)$), and it follows from Lemma 1.98 that

$$r^{-1}|\cos\theta|\,\|R(re^{i\theta}; B)\| \leq r^{-2}, \tag{4}$$

and therefore the left-hand side of (4) converges to zero uniformly with respect to θ. By the corollary to Lemma 3.13.1 in [HP], $R(\cdot; B)$ is a polynomial of degree < 1, i.e., a constant, and since $\|R(\lambda; B)\| < 1/\lambda \to 0$ for $0 < \lambda \to \infty$, we get the absurd conclusion $R(\cdot; B) = 0$ (this would imply that $y = (\lambda I - B)R(\lambda; B)y = 0$ for all $y \in Y$, i.e., $Y = \{0\}$, contrary to our hypothesis). □

E.6 The ABLV Stability Theorem

We now return to our particular C_o-semigroup of isometries $U(\cdot)$ induced by the *nonstable* C_o-semigroup of contractions $T(\cdot)$ on the nontrivial asymptotic space Y of $T(\cdot)$. If Case (a) in Lemma 1.100 holds, then by (13) in Section E.4 (and the fact that $\sigma(A)$ is contained in the closed left halfplane, since A generates the C_o-semigroup of contractions $T(\cdot)$), we conclude that $\sigma(A)$ is *equal* to the closed left halfplane, hence in particular $\sigma(A) \cap i\mathbb{R} = i\mathbb{R}$. Since we are looking for sufficient conditions for *stability* of $T(\cdot)$, we shall *exclude* the latter possibility by making the following

Hypothesis I. $\sigma(A) \cap i\mathbb{R}$ *is countable.*

(We write "countable" for the more precise expression "at most countable.")

Thus Case (b) holds. Hence, by (13) in Section E.4, $\sigma(B)$ is a (nonempty, closed) subset of $\sigma(A) \cap i\mathbb{R}$, and is therefore countable (by Hypothesis I). However, as a nonempty closed subset of $\mathbb{C}$, it is a (nonempty) complete metric space, and is therefore of Baire's Second Category in itself. Since it is a countable union of singletons (which are closed sets in it), at least one of these singletons, say $\{\lambda_0\}$), contains a nonempty (relatively) open set. Thus there exists an open set $V \subset \mathbb{C}$ such that $\emptyset \neq V \cap \sigma(B) \subset \{\lambda_0\}$, hence $V \cap \sigma(B) = \{\lambda_0\}$. This means that λ_0 is an *isolated point* of $\sigma(B)$.

We wish to use the Riesz projection associated with an isolated point of the spectrum of a *bounded* operator (cf. Construction 9.22 in [K17]). We then shift to the resolvent $R(\alpha; B)$ for some fixed $\alpha \in \rho(B)$, and set $h(\lambda) := (\alpha - \lambda)^{-1}$. By Theorem 10.2 in [K17], h is a homeomorphism of $\sigma(B) \cup \{\infty\}$ onto $\sigma(R(\alpha; B))$. Thus $\mu_0 := h(\lambda_0)$ is an isolated point of $\sigma(R(\alpha; B))$. Let P be the Riesz projection associated with μ_0 for the bounded operator $R(\alpha; B)$ (cf. Construction 9.22 in [K17]). Then $P \neq 0$ reduces $R(\alpha; B)$ and $\sigma(R(\alpha; B)|_{PY}) = \{\mu_0\}$. It follows that $\alpha \in \rho(B|_{PY})$ and

$$R(\alpha; B|_{PY}) = R(\alpha; B)|_{PY}.$$

Hence

$$\sigma(R(\alpha; B|_{PY})) = \{\mu_0\} \neq \{0\},$$

and therefore $R(\alpha; B|_{PY})^{-1}$ is a bounded operator on PY. Consequently $B|_{PY} = \alpha I - R(\alpha; B|_{PY})^{-1}$ *is a bounded operator on* PY. In particular, $\sigma(B|_{PY}) \neq \emptyset$. But

$$\sigma(B|_{PY}) \subset h^{-1}[\sigma(R(\alpha; B|_{PY}))] = \{h^{-1}(\mu_0)\} = \{\lambda_0\},$$

hence $\sigma(B|_{PY}) = \{\lambda_0\}$.

Lemma 1.101 (Under Hypothesis I, $T(\cdot)$ not stable, and notation as above). $U(t)|_{PY} = e^{\lambda_0 t} I$ *and* $B|_{PY} = \lambda_0 I$.

Proof. $V(t) := U(t)|_{PY}$, $t \in \mathbb{R}$, is a C_o-group of isometries with the *bounded* generator $B|_{PY}$. By Theorem 1.5, $V(t) = \exp(t\,B|_{PY})$. By the Spectral Mapping Theorem, we then have $\sigma(V(t)) = \exp(t\sigma(B|_{PY})) = \{\exp(t\lambda_0)\}$. Consider the C_o-group of isometries $W(t) := e^{-\lambda_0 t} V(t)$ on PY (recall that $\lambda_0 \in i\mathbb{R}$). Fix an arbitrary $t \in \mathbb{R}$, and write $W := W(t)$. Since $\sigma(W) = e^{-\lambda_0 t}\sigma(V(t)) = \{1\}$, the function $f(z) = -i \log z$ is analytic in a neighborhood of $\sigma(W)$, and therefore $S := f(W)$ is well-defined by means of the analytic operational calculus for W. If $g(z) := e^{iz}$, then since $(g \circ f)(z) = z$, we have by the Composite Function Theorem for the analytic operational calculus (cf. [K17, Theorem 9.21]):

$$W = (g \circ f)(W) = g(f(W)) = g(S) = e^{iS}. \tag{1}$$

By the Spectral Mapping Theorem for the analytic operational calculus (cf. [K17, Theorem 9.20]),

$$\sigma(S) = \sigma(f(W)) = f(\sigma(W)) = \{f(1)\} = \{0\}. \tag{2}$$

For any $n \in \mathbb{N}$, consider the operator $\sin(nS)$ (defined by means of the analytic operational calculus with the entire function $\sin(nz)$). Then by (1) (and the fact that $W(\cdot)$ is a group of isometries),

$$\begin{aligned} \|\sin(nS)\| &= \|(1/2i)[e^{inS} - e^{-inS}]\| = (1/2)\|W^n - W^{-n}\| \\ &\le (1/2)[\|W(nt)\| + \|W(-nt)\|] = 1. \end{aligned} \tag{3}$$

By the Spectral Mapping Theorem and (2),

$$\sigma(\sin(nS)) = \sin(n\sigma(S)) = \{0\}.$$

Consider the principal branch of $\arcsin z$. It is analytic at 0, and its power series expansion in the unit disk, $\sum a_k z^k$, has *positive* coefficients with sum $\sum a_k = \arcsin 1 = \pi/2$. Therefore, by the Composite Function Theorem and (3),

$$\begin{aligned} n\,\|S\| = \|\arcsin(\sin(nS))\| &= \left\| \sum_k a_k [\sin(nS)]^k \right\| \\ &\le \sum_k a_k \|\sin(nS)\|^k \le \sum a_k = \pi/2 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $S = 0$, and therefore $W = I$ (by (1)). Since $t \in \mathbb{R}$ was arbitrary, we proved that $W(t) = I$ for all $t \in \mathbb{R}$, that is,

$$U(t)|_{PY} = e^{\lambda_0 t} I \quad (t \in \mathbb{R}). \tag{4}$$

The generator of this group is trivially $B|_{PY} = \lambda_0 I$. □

Lemma 1.102 (Assumptions as in Lemma 1.101; same notation). *λ_0 is an eigenvalue of A^*.*

Proof. Since $PY \neq \{0\}$, there exists $z^* \in (PY)^*$, $z^* \neq 0$. Define x^* by

$$x^* x = z^*[P(\pi x)] \quad (x \in X). \tag{5}$$

Clearly $x^* \in X^*$. If $x^* = 0$, then since $z^* \circ P$ is continuous on Y and πX is dense in Y, it follows from (5) that $z^* P y = 0$ for all $y \in Y$, i.e., z^* is the zero functional on PY, contrary to our choice. Hence $x^* \neq 0$.

For all $x \in D(A)$, we have by (5)–(6) in Section E.4, and Lemma 1.101:

$$x^*Ax = z^*P\pi(Ax) = z^*PB(\pi x) = z^*BP(\pi x) = z^*(\lambda_0 I)P(\pi x)$$
$$= \lambda_0 z^* P(\pi x) = \lambda_0 x^* x.$$

In particular, it follows that the map $x \in D(A) \to x^*Ax$ is continuous, that is, $x^* \in D(A^*)$; and furthermore, $A^*x^* = \lambda_0 x^*$. Since $x^* \neq 0$, this shows that λ_0 is an eigenvalue of A^* (and x^* is an associated eigenvector). □

Since $\lambda_0 \in i\mathbb{R}$, Lemma 1.102 yields a (desired!) contradiction if we make the following

Hypothesis II. $P\sigma(A^*) \cap i\mathbb{R} = \emptyset$.

(Recall that $P\sigma(\cdot)$ denotes the point spectrum.)

When this is the case, the last contradiction disproves our initial assumption that $T(\cdot)$ is *not stable*.

Finally, we drop the assumption that $T(\cdot)$ is a *contraction* semigroup by using the *equivalent* norm

$$|x| = \sup_{t \geq 0} \|T(t)x\| \quad (x \in X).$$

(Clearly, $\|x\| \leq |x| \leq M\,\|x\|$ for all $x \in X$, where $M := \sup_{t\geq 0} \|T(t)\|$.)

With respect to the norm $|\cdot|$, $T(\cdot)$ is a C_o-semigroup of *contractions*. Since the norms are equivalent, the renorming does not change the generator A, its spectrum, the point spectrum of A^*, and the stability status of $T(\cdot)$. Therefore our previous conclusion regarding contraction semigroups remains valid (through the said renorming) for any bounded C_o-semigroup. We thus proved the following result, due to Arendt, Batty, Lyubich, and Vu (the "ABLV stability theorem"):

Theorem 1.103 (ABLV Stability Theorem). *Let $T(\cdot)$ be a bounded C_o-semigroup, and let A be its generator. Assume*

I. $\sigma(A) \cap i\mathbb{R}$ *is countable; and*
II. $P\sigma(A^*) \cap i\mathbb{R} = \emptyset$.

Then $T(\cdot)$ is stable.

F

Boundary Values

Recall that the C_o-semigroup $T(\cdot)$ on the Banach space X is said to be analytic (or *holomorphic*) in $\mathbb{C}^+$ if it extends to an analytic $B(X)$-valued function $W(\cdot)$ on $\mathbb{C}^+$, and the C_o-property is strengthened to the requirement

$$\lim_{z\in\mathbb{C}^+;\, z\to 0} W(z) = I \tag{1}$$

in the s.o.t. (cf. Definition 1.53).

Necessarily, the analyticity of $W(\cdot)$ and the semigroup property of its restriction $T(\cdot)$ imply that $W(\cdot)$ satisfies the extended semigroup property

$$W(z)W(w) = W(z+w) \quad (z, w \in \mathbb{C}^+). \tag{2}$$

In this section, we are interested in the existence of "boundary values" of the extension $W(\cdot)$, that is, the existence of the limits

$$\lim_{s\to 0+} W(s+it) := U(t) \tag{3}$$

in the s.o.t., for all $t \in \mathbb{R}$.

By the Uniform Boundedness Theorem, a necessary condition for the existence of boundary values of $W(\cdot)$ is the boundedness of $\|W(\cdot)\|$ on the "unit rectangle"

$$Q := \{z = x + iy;\, 0 < x \leq 1,\ -1 \leq y \leq 1\}.$$

The fact that this condition is also sufficient is part of the statement of Theorem 1.105 below.

F.1 Regular Semigroups and Boundary Values

We begin with a convenient definition.

Definition 1.104. *A regular semigroup is a C_o-semigroup $T(\cdot)$ which admits an analytic extension $W(\cdot)$ to $\mathbb{C}^+$, such that $\|W(\cdot)\|$ is bounded on the unit rectangle Q.*

S. Kantorovitz, *Topics in Operator Semigroups*,
Progress in Mathematics 281, DOI 10.1007/978-0-8176-4932-6_6,

(We shall occasionally say in this case that *the extension* $W(\cdot)$ is a regular semigroup.)

We shall also fix the notation (for a given regular semigroup $W(\cdot)$)

$$b := \sup_{z \in Q} \log \|W(z)\|. \tag{4}$$

Clearly $0 \leq b < \infty$.

Theorem 1.105. *Let $T(\cdot)$ be a regular semigroup, and let A be its generator. Denote its analytic extension to $\mathbb{C}^+$ by $W(\cdot)$. Then*

(a) *the boundary values $U(\cdot)$ defined by (3) exist (in the s.o.t.).*
(b) *$U(\cdot)$ is a C_o-group (called the "boundary group" associated with the regular semigroup $T(\cdot)$); it commutes with $T(\cdot)$ and with $W(\cdot)$.*
(c) *$W(s+it) = T(s)U(t)$ for all $s > 0$ and $t \in \mathbb{R}$, and satisfies (1).*
(d) *$\|U(t)\| \leq N\, e^{b|t|} \quad (t \in \mathbb{R})$, for a suitable constant $N \geq 1$.*
(e) *The generator of $U(\cdot)$ is iA.*

Conversely, *let $T(\cdot)$ be a C_o-semigroup, let A be its generator, and suppose that iA generates a C_o-group $U(\cdot)$. Then $T(\cdot)$ is a regular semigroup, $U(\cdot)$ is the boundary group associated with it, and the (unique) analytic extension of $T(\cdot)$ to $\mathbb{C}^+$ is given by (c).*

Proof. Statements (a) and (d). Fix $t \in \mathbb{R}$, and let $n \in \mathbb{N} \cup \{0\}$ be such that $n \leq |t| < n+1$. For $0 < s \leq 1$, we have $(s+it)/(n+2) \in Q$, hence

$$\|W(s+it)\| = \left\| \left[W\left(\frac{s+it}{n+2} \right) \right]^{n+2} \right\| \leq e^{b(n+2)} \leq e^{b(2+|t|)} = N\, e^{b|t|}, \tag{5}$$

for all $0 < s \leq 1$ and $t \in \mathbb{R}$, where $N := e^{2b}$.

Fix $x \in X$. For any $h > 0$, the strong continuity of $W(\cdot)$ in $\mathbb{C}^+$ (which follows from the analyticity of $W(\cdot)$ in $\mathbb{C}^+$) implies that

$$W(s+it)T(h)x = W(s+h+it)x \to W(h+it)x \tag{6}$$

as $s \to 0+$. Let $\epsilon > 0$ be given. By the C_o-property of $T(\cdot)$, we may fix $h > 0$ such that

$$\|x - T(h)x\| < (\epsilon/4)e^{-b(|t|+2)}.$$

For this fixed h, it follows from (6) that there exists $\delta \in (0,1]$ such that

$$\|W(s+it)T(h)x - W(s'+it)T(h)x\| < \epsilon/2 \tag{7}$$

for $0 < s, s' < \delta$. Then for these s, s', we have by (5) and (7)

$$\begin{aligned} &\|W(s+it)x - W(s'+it)x\| \\ &\quad \le \|W(s+it)T(h)x - W(s'+it)T(h)x\| \\ &\qquad + \Big[\|W(s+it)\| + \|W(s'+it)\|\Big]\|x - T(h)x\| < \epsilon. \end{aligned}$$

This proves (a). Also $U(t) \in B(X)$ and (d) follows from (5) (for all $t \in \mathbb{R}$) by a well-known consequence of the Uniform Boundedness Theorem.

Statements (b) and (c). For $x \in X$, $s > 0$, and $t, t' \in \mathbb{R}$ fixed, we have by (a), the semigroup property of $W(\cdot)$, and the fact that $W(s+it) \in B(X)$:

$$\begin{aligned} U(t')W(s+it)x &= \lim_{s'\to 0+} W(s'+it')W(s+it)x \\ &= \lim_{s'\to 0+} W(s+it)W(s'+it')x \\ &= W(s+it) \lim_{s'\to 0+} W(s'+it')x = W(s+it)U(t')x. \qquad (8) \end{aligned}$$

Thus $U(t')$ commutes with $W(s+it)$, hence in particular with $T(s)$ (case $t = 0$), for all $t' \in \mathbb{R}$ and $s > 0$.

Also by (8), together with the semigroup property and the strong continuity of $W(\cdot)$ in $\mathbb{C}^+$,

$$W(s+it)U(t')x = \lim_{s'\to 0+} W(s+s'+i(t+t'))x = W(s+i(t+t'))x.$$

Letting $s \to 0+$, it follows from (a) that

$$U(t)U(t')x = U(t+t')x \quad (t, t' \in \mathbb{R};\ x \in X),$$

that is, $U(\cdot)$ is a group of operators on X.

For $s > 0$, $t \in \mathbb{R}$, and $x \in X$ given, we have

$$\begin{aligned} T(s)U(t)x = T(s) \lim_{s'\to 0+} W(s'+it)x &= \lim_{s'\to 0+} T(s)W(s'+it)x \\ = \lim_{s'\to 0+} W(s+s'+it)x &= W(s+it)x, \end{aligned}$$

where we used the strong continuity of $W(\cdot)$ in $\mathbb{C}^+$. Thus $W(s+it) = T(s)U(t)$ for all $s > 0$ and $t \in \mathbb{R}$.

Again by strong continuity of $W(\cdot)$ and (c), for any $t, h > 0$ and $x \in X$

$$[U(t) - I]T(h)x = W(h+it)x - W(h)x \to 0 \qquad (9)$$

when $t \to 0$.

Given $\epsilon > 0$ and $x \in X$, fix $h > 0$ such that

$$\|x - T(h)x\| < (\epsilon/2)[1 + e^{3b}]^{-1}$$

(by the C_o-property of $T(\cdot)$). For this h, it follows from (9) that we may choose $\delta \in (0, 1]$ such that

$$\|[U(t) - I]T(h)x\| < \epsilon/2$$

for $0 < t < \delta$. Then for these values of t, we have by (5)

$$\|[U(t) - I]x\| \leq \|[U(t) - I]T(h)x\| + (1 + e^{3b})\|x - T(h)x\| < \epsilon.$$

Thus $U(\cdot)$ has the C_o-property. Finally, for any $x \in X$ and $s + it \in \mathbb{C}^+$, we have by Theorem 1.1:

$$\begin{aligned}\|W(s + it)x - x\| &\leq \|T(s)[U(t)x - x]\| + \|T(s)x - x\| \\ &\leq M\, e^{as}\|U(t)x - x\| + \|T(s)x - x\| \to 0\end{aligned}$$

as $s+it \to 0$ (in $\mathbb{C}^+$), by the C_o-property of $T(\cdot)$ and $U(\cdot)$. Thus $W(\cdot)$ satisfies (1), and the proof of (b) and (c) is complete.

Statement (e). Denote by B the generator of the C_o-group $U(\cdot)$. Let $x \in D(A)$. By the analyticity of $W(\cdot)$ in $\mathbb{C}^+$, (c), Theorem 1.2, and the Cauchy–Riemann equation, we have for all $s > 0$ and $t \in \mathbb{R}$:

$$\begin{aligned}U(t)T(s)Ax &= \frac{\partial}{\partial s}[U(t)T(s)x] = \frac{\partial}{\partial s}[W(s + it)x] \\ &= -i\frac{\partial}{\partial t}W(s + it)x = -i\frac{\partial}{\partial t}U(t)[T(s)x].\end{aligned}$$

Taking this identity at $t = 0$, we conclude that $T(s)x \in D(B)$ and

$$B\,T(s)x = iT(s)Ax \quad (s > 0). \tag{10}$$

When $s \to 0+$, $T(s)x \to x$, and by (10), $B\,T(s)x \to iAx$. Since B is closed, it follows that $x \in D(B)$ and $Bx = iAx$. Hence $iA \subset B$.

On the other hand, if $x \in D(B)$, we get similarly

$$\begin{aligned}-iT(s)U(t)Bx &= -i\frac{\partial}{\partial t}[W(s + it)x] = \frac{\partial}{\partial s}[W(s + it)x] \\ &= \frac{\partial}{\partial s}T(s)[U(t)x]\end{aligned}$$

for all $s > 0$ and $t \in \mathbb{R}$. Equivalently,

$$h^{-1}[T(s + h)U(t)x - T(s)U(t)x] \to -iT(s)U(t)Bx$$

as $h \to 0+$, that is, as $h \to 0+$,

$$h^{-1}[T(h) - I]\, T(s)U(t)x \to -iT(s)U(t)Bx \quad (s > 0,\, t \in \mathbb{R}).$$

Thus $T(s)U(t)x \in D(A)$ and

$$A[T(s)U(t)x] = -iT(s)U(t)Bx. \tag{11}$$

In particular (for $t = 0$), $T(s)x \in D(A)$ and $AT(s)x = -iT(s)Bx$ for all $s > 0$. As $s \to 0+$, $T(s)x \to x$ and $AT(s)x \to -iBx$. Since A is closed, it follows that $x \in D(A)$ and $Ax = -iBx$, and we conclude that $B \subset iA$. Hence $B = iA$.

Converse statement. By Theorem 1.1 applied to the C_o-semigroups $\{T(s); s \geq 0\}$, $\{U(t); t \geq 0\}$, and $\{U(-t); t \geq 0\}$, there exist constants $a, b \geq 0$ and $M, N \geq 1$ such that

$$\|T(s)\| \leq Me^{as}; \quad \|U(t)\| \leq Ne^{b|t|} \quad (s \geq 0;\, t \in \mathbb{R}). \tag{12}$$

(Presently, the constants N, b are not related to the preceding constants with the same notation.)

We *define* $W(\cdot)$ on $\mathbb{C}^+$ by relation (c).

Fix $x \in D(A)$, and consider the X-valued function

$$g(s + it) := W(s + it)Ax = T(s)U(t)Ax \quad (s > 0,\, t \in \mathbb{R}). \tag{13}$$

Since iA is by hypothesis the generator of the C_o-group $U(\cdot)$, we also have (by Theorem 1.2)

$$g(s + it) = T(s)AU(t)x \quad (s > 0,\, t \in \mathbb{R}).$$

Observe that g is strongly continuous in $\mathbb{C}^+$, because for all $s, s' > 0$ and $t, t' \in \mathbb{R}$,

$$\|g(s + it) - g(s' + it')\| \leq \|[T(s) - T(s')][U(t)Ax]\| \\ + \|T(s')[U(t) - U(t')]\, Ax\|;$$

when $s' + it' \to s + it$, the first term on the right-hand side has limit 0 by the strong continuity of $T(\cdot)$, and by (12), the second term is $\leq M\, e^{as'} \|[U(t) - U(t')]\, Ax\| \to 0$, by the strong continuity of $U(\cdot)$ (cf. Theorem 1.1).

Next, it follows from the definition of $W(\cdot)$ and Theorem 1.2 that $U(t)x \in D(iA) = D(A)$ for all $t \in \mathbb{R}$ and

$$\frac{\partial}{\partial s} W(s + it)x = T(s)AU(t)x \\ = -iT(s)(iA)U(t)x = -i\frac{\partial}{\partial t} W(s + it)x \tag{14}$$

for all $s + it \in \mathbb{C}^+$.

Thus, for each $x \in D(A)$ and $x^* \in X^*$, the complex-valued function $x^*W(\cdot)x$ satisfies the Cauchy–Riemann equation and has continuous partial derivatives (by our observation on the function g) in $\mathbb{C}^+$. Therefore, by (the classical) Theorem 11.2 in [R1], $x^*W(\cdot)x$ is analytic in $\mathbb{C}^+$ (for all $x \in D(A)$ and $X^* \in X^*$).

Fix $x^* \in X^*$ and $x \in X$, $x \neq 0$. Since $D(A)$ is dense in X, we may choose $x_n \in D(A)$ such that $x_n \to x$ strongly in X and $\|x_n\| \le 2\,\|x\|$ for all $n \in \mathbb{N}$. For any compact subset H of $\mathbb{C}^+$, we have by (12)

$$\begin{aligned}|x^*W(s+it)x_n| &\le 2M\,N\,e^{as+b|t|}\|x^*\|\,\|x\| \\ &\le 2M\,N\,e^{a\sigma+b\tau}\|x^*\|\,\|x\| \end{aligned} \tag{15}$$

for all $s+it \in H$, where $\sigma = \sigma(H) := \sup_{z\in H}\Re z$, and $\tau = \tau(H) := \sup_{z\in H}|\Im z|$. Since $x_n \in D(A)$, it follows that $\{x^*W(\cdot)x_n\}$ is a family of analytic functions in $\mathbb{C}^+$, uniformly bounded on each compact subset of $\mathbb{C}^+$. It is therefore a *normal family* (cf. Theorem 14.6 in [R1]). Let then $\{x^*W(\cdot)x_{n_k}\}$ be a subsequence converging uniformly on compact subsets of $\mathbb{C}^+$. The limit function, $x^*W(\cdot)x$, is then analytic in $\mathbb{C}^+$ (cf. Theorem 10.27 in [R1]). Since this is true for all $x \in X$ and $x^* \in X^*$, it follows from Theorem 3.10.1 in [HP] that the operator-valued function $W(\cdot)$ is analytic in $\mathbb{C}^+$. For all $s > 0$, we have $W(s) = T(s)U(0) = T(s)$, so that $W(\cdot)$ is indeed an analytic extension of $T(\cdot)$ to $\mathbb{C}^+$, and by (12)

$$\sup_{z\in Q}\|W(z)\| \le \sup_{0<s\le 1}\|T(s)\|\sup_{|t|\le 1}\|U(t)\| \le M\,N\,e^{a+b} < \infty.$$

Hence $T(\cdot)$ ia a regular semigroup, $W(\cdot)$ is its (unique) analytic extension to $\mathbb{C}^+$, and the associated boundary group is clearly $U(\cdot)$ (because $\lim_{s\to 0+} W(s+it)x = \lim_{s\to 0+} T(s)[U(t)x] = U(t)x$, by the C_o-property of $T(\cdot)$). □

The "converse" part of Theorem 1.104 can be restated as a solution of the extension problem of a C_o-*group* $U(\cdot)$ to an analytic semigroup in $\mathbb{C}^+$, whose boundary group is the given group $U(\cdot)$:

Theorem 1.106. *Let $U(\cdot)$ be a C_o-group, and let iA be its generator. Then $U(\cdot)$ is the boundary group associated with a regular semigroup if and only if A generates a C_o-semigroup, $T(\cdot)$.*

When this is the case, $T(\cdot)$ is the unique regular semigroup with associated boundary group $U(\cdot)$, and the unique analytic extension of $U(\cdot)$ to $\mathbb{C}^+$ is $W(s+it) = T(s)U(t)$, $s > 0$, $t \in \mathbb{R}$.

F.2 The Generator of a Regular Semigroup

We shall apply Theorem 1.105 to obtain various characterizations of the generator of a regular semigroup.

Corollary 1.107. *Let A be an operator on the Banach space X with domain $D(A)$. Then A generates a regular semigroup if and only if the following conditions (a)–(c) are satisfied:*

(a) *$D(A)$ is dense in X;*
(b) *the resolvent set of A contains the rays (a,∞) and $\pm i(b,\infty)$ for some $a, b \geq 0$;*
(c) *one has*

$$\sup_{s>a;\, n\in\mathbb{N}} \|[(s-a)R(s;A)]^n\| < \infty$$

and

$$\sup_{t>b,\, n\in\mathbb{N}} \|[(t-b)R(\pm it;A)]^n\| < \infty.$$

Proof. If A generates a regular semigroup $T(\cdot)$, then by Theorem 1.105, iA generates a C_o-group (the boundary group associated with $T(\cdot)$); equivalently, iA and $-iA$ generate C_o-semigroups. Therefore Conditions (a)–(c) follow from the necessary part of the Hille–Yosida theorem (Theorem 1.17) and the relation

$$R(\lambda; \pm iA) = \mp iR(\mp i\lambda; A) \tag{1}$$

(whenever either side makes sense).

Conversely, suppose A satisfies Conditions (a)–(c). By (a), (b), and the first condition in (c), it follows from the Hille–Yosida theorem that A generates a C_o-semigroup. By (1), the second condition in (c) may be written in the form

$$\sup_{t>b;\, n\in\mathbb{N}} \|[(t-b)R(\pm t; iA)]^n\| < \infty.$$

Together with (a) and (b), this implies that the operator iA generates a C_o-group (cf. Theorem 1.39). By Theorem 1.105, we conclude that A generates a regular semigroup. □

Corollary 1.108. *Let $T(\cdot)$ be a C_o-semigroup of contractions, and let A be its generator. Then $T(\cdot)$ extends to an analytic semigroup of contractions in $\mathbb{C}^+$ if and only if iA generates a C_o-group of contractions.*

Proof. If $T(\cdot)$ extends to an analytic semigroup of contractions in $\mathbb{C}^+$, it is trivially regular, and therefore iA generates the associated boundary group (by Theorem 1.105), which is necessarily a C_o-group of contractions. Conversely, if iA generates a C_o-group of contractions $U(\cdot)$, then by Theorem 1.105, $T(\cdot)$ is regular, and its analytic extension in $\mathbb{C}^+$ is $W(s+it) = T(s)U(t)$, which consists clearly of contractions. □

Remark 1.109. In Theorem 1.54, we considered the more general case of extending $T(\cdot)$ as an analytic semigroup of contractions in an *arbitrary* sector

$$S_\theta := \{z = re^{i\phi};\, r > 0,\, -\theta < \phi < \theta\},$$

where $0 < \theta \leq \pi/2$. The criterion in that theorem is that $e^{i\alpha}A$ generate a C_o-semigroup for *all* $\alpha \in (-\theta, \theta)$. Here, *in the special case* $\theta = \pi/2$, the criterion obtained in Corollary 1.108 involves only the endpoint values $\alpha = \pm\pi/2$ (namely, that $\pm iA$ generate C_o-semigroups of contractions).

Corollary 1.110. *The operator A with domain $D(A)$ in the Banach space X generates an analytic contraction semigroup if and only if the following two conditions are satisfied:*

(a) *$D(A)$ is dense in X;*
(b) *$sR(s; A)$ and $tR(\pm it; A)$ exist and are contractions for all $s, t > 0$.*

Proof. This follows from Corollaries 1.18 and 1.108, and Relation (1). □

Corollary 1.111. *Let A generate a regular semigroup on the Banach space X. Then $A + B$ generates a regular semigroup for any $B \in B(X)$.*

Proof. By Theorem 1.38 and the necessity part of Theorem 1.105, the operators $A + B$ and $i(A + B)$ $(= (iA) + (iB))$ generate a C_o-semigroup and a C_o-group, respectively. Therefore $A + B$ generates a regular semigroup, by the "converse" part of Theorem 1.105. □

Recall that the *numerical range* of an operator B is the set

$$\nu(B) := \{x^*Bx;\ x \in D(B),\, x^* \in X^*,\, \|x\| = \|x^*\| = x^*x = 1\}.$$

(Cf. Definition 1.24.)

Corollary 1.112. *Let A generate an analytic C_o-semigroup of contractions in $\mathbb{C}^+$. Let B be an operator satisfying the following two conditions:*

(a) *$\nu(B) \subset (-\infty, 0]$;*
(b) *$D(A) \subset D(B)$ and there exist constants $0 \leq a < 1$ and $b \geq 0$ such that*

$$\|Bx\| \leq a\,\|Ax\| + b\,\|x\| \quad (x \in D(A)).$$

Then $A + B$ generates an analytic C_o-semigroup of contractions in $\mathbb{C}^+$.

Proof. By Theorem 1.30, Conditions (a) and (b) imply that $A + B$ generates a C_o-semigroup of contractions. By Corollary 1.108, iA and $-iA$ generate C_o-semigroups of contractions. Condition (a) implies that $\Re\nu(\pm iB) = \mp\Im\nu(B) = \{0\}$, and therefore the operators $\pm iB$ are trivially dissipative. They are also $\pm iA$-bounded with $\pm iA$-bound < 1 (by Condition (b)) (cf. Definitions 1.25 and 1.28 for the terminology). Consequently, by Theorem 1.30, the operators $i(A + B)$ and $-i(A + B)$ generate C_o-semigroups of contractions. We now conclude from Corollary 1.108 that $A + B$ generates an analytic C_o-semigroup of contractions. □

Corollary 1.113. *Let $T(\cdot)$ be a regular semigroup. Let $W(\cdot)$ be its analytic extension to $\mathbb{C}^+$, and denote $K := \sup_Q \|W(\cdot)\|$ $(< \infty)$. Then*

$$\|T(s)\| \leq K^{5/2}\|T(1)\|^s \quad (s > 0).$$

Proof. By Theorem 1.105, $W(s+it) = T(s)U(t)$, where $U(\cdot)$ denotes the associated boundary group of $T(\cdot)$, and $\|U(t)\| \leq N\, e^{b|t|}$ (where $b := \log K$, and by the proof of Part (a), we may choose $N = e^{2b} = K^2$).

Consider the operator-valued strongly continuous function

$$\Phi(z) = e^{bz^2}W(z)$$

on $\overline{\mathbb{C}^+}$ (where $\Phi(it) := e^{-bt^2}U(t)$). It is analytic in $\mathbb{C}^+$, and

$$\begin{aligned}\|\Phi(s+it)\| &= e^{b(s^2-t^2)}\|T(s)\|\,\|U(t)\| \leq K^2 \exp(b\,[s^2 - t^2 + |t|])\|T(s)\| \\ &\leq K^2 \exp(b\,[s^2 + 1/4])\|T(s)\| \qquad (2)\end{aligned}$$

(because $-t^2 + |t| = |t|(1-|t|) \leq 1/4$). By (2), Φ is bounded in each vertical strip $\{s+it; k-1 \leq s \leq k,\, t \in \mathbb{R}\}$. Also for each $n = 0, 1, 2, \ldots$ and $t \in \mathbb{R}$,

$$\|\Phi(n+it)\| \leq K^2 \exp(b\,[n^2 + 1/4])\|T(1)\|^n. \qquad (3)$$

If $s \in (k-1, k]$, write $s = p(k-1) + (1-p)k = k - p$ with $p \in [0,1)$. By the "Three Lines Theorem" for operator-valued functions (cf. Theorem VI.10.3 in [DS I–III]), it follows from (3) that

$$\begin{aligned}\|\Phi(s+it)\| &\leq K^2 e^{b/4} \exp(b[p(k-1)^2 + (1-p)k^2])\|T(1)\|^s \\ &\leq K^2 \exp(b[s^2 + 1/2])\|T(1)\|^s \qquad (4)\end{aligned}$$

(because $p(k-1)^2 + (1-p)k^2 = (k-p)^2 + p(1-p) = s^2 + p(1-p) \leq s^2 + 1/4$, and $b \geq 0$). Taking $t = 0$, it follows from (2) and (4) that for all $s > 0$

$$e^{bs^2}\|T(s)\| \leq K^{5/2}e^{bs^2}\|T(1)\|^s. \qquad \square$$

F.3 Examples of Regular Semigroups

We shall discuss now some classical examples of regular semigroups. The same method is essentially used in all examples, but we preferred to avoid general statements, and to proceed in a rather leasurely manner, allowing for some amount of repetition.

Example 1.114. The Gauss–Weierstrass semigroup.

For $t > 0$ and $x \in \mathbb{R}^n$, let

$$k_t(x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), \tag{1}$$

where $|x|$ denotes the Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$|x| := \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}.$$

Clearly k_t belongs to the *Schwartz space* $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ of *rapidly decreasing functions*, and it is well-known that its Fourier transform is

$$\hat{k}_t(u) = e^{-t|u|^2} \quad (u \in \mathbb{R}^n) \tag{2}$$

(cf. [K17, Section I.3.12]).

In particular, since $k_t \geq 0$,

$$\|k_t\|_1 := \|k_t\|_{L^1(\mathbb{R}^n)} = \hat{k}_t(0) = 1. \tag{3}$$

If $s, t > 0$, it follows from (2) that the Fourier transform of the convolution $k_t * k_s$ is equal to

$$\hat{k}_t(u)\,\hat{k}_s(u) = \exp[-(t+s)\,|u|^2] = \hat{k}_{t+s}(u),$$

and therefore, by the uniqueness property of the Fourier transform,

$$k_t * k_s = k_{t+s} \quad (t, s > 0). \tag{4}$$

Define the operator $T(t)$ on $L^p := L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ by

$$T(t)f = k_t * f \quad (t > 0;\ f \in L^p); \tag{5}$$
$$T(0) = I$$

(where I denotes the identity operator on L^p).

We have

$$\|T(t)f\|_p \leq \|k_t\|_1 \|f\|_p = \|f\|_p,$$

that is, $T(t)$ is a *contraction operator* on L^p for all $t \geq 0$.

If $f \in \mathcal{S}$ and $s, t > 0$, the associativity of convolution on $\mathcal{S}$ and (4) imply that

$$T(t)T(s)f = k_t * (k_s * f) = (k_t * k_s) * f = k_{t+s} * f = T(t+s)f.$$

Since $\mathcal{S}$ is dense in L^p and the operators $T(t)T(s)$ and $T(t+s)$ are in $B(L^p)$, it follows that $T(\cdot)$ satisfies the semigroup relation on L^p.

For simplicity, we shall restrict now our discussion to the case $p = 2$, unless stated otherwise.

Since $\mathcal{F} : f \in \mathcal{S} \to (2\pi)^{-n/2}\hat{f} \in \mathcal{S}$ preserves the L^2 norm (cf. for example [K17, Section II.3.1, Relation (19)]), we have by (2), for all $f \in \mathcal{S}$ and $t > 0$,

$$\|T(t)f - f\|_2 = \|k_t * f - f\|_2 = (2\pi)^{-n/2}\|\hat{k}_t \hat{f} - \hat{f}\|_2$$
$$= (2\pi)^{-n/2}\|(e^{-t|u|^2} - 1)\hat{f}(u)\|_2.$$

The functions in the last norm sign are dominated by $2|\hat{f}| \in \mathcal{S} \subset L^2$ and converge to 0 as $t \to 0$. By the Dominated Convergence Theorem, we conclude that $T(t)f \to f$ in L^2-norm for all $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in L^2 and $\|T(t) - I\|_{B(L^2)} \le 2$, it follows that $T(t) \to I$ in the s.o.t. on L^2, when $t \to 0+$.

Thus $T(\cdot)$ is a C_o-semigroup on L^2 (the so-called *Gauss–Weierstrass semigroup*).

Consider now the "kernel" k_z (with the positive parameter t replaced by the complex parameter $z \in \mathbb{C}^+$):

$$k_z(x) := (4\pi z)^{-n/2} \exp\left(-\frac{|x|^2}{4z}\right),$$

where the branch of $z^{1/2}$ is chosen such that $z^{1/2}$ is positive for z positive. Since

$$|k_z(x)| = (4\pi|z|)^{-n/2} \exp\left(-\frac{\Re z}{4|z|^2}\,|x|^2\right),$$

$k_z \in \mathcal{S}$ for each fixed $z \in \mathbb{C}^+$, and therefore, for each fixed $u \in \mathbb{R}^n$, the integral

$$F_u(z) := \int_{\mathbb{R}^n} e^{-iu\cdot x} k_z(x)\, dx$$

converges absolutely ($u \cdot x$ denotes the inner product on $\mathbb{R}^n$). It follows that F_u is analytic in $\mathbb{C}^+$ (for each fixed $u \in \mathbb{R}^n$), and coincides with the analytic function $e^{-z|u|^2}$ for $z \in (0,\infty)$ (by (2)). Hence

$$\hat{k}_z(u) = e^{-z|u|^2} \quad (u \in \mathbb{R}^n;\ z \in \mathbb{C}^+). \tag{6}$$

Define

$$W(z)f := k_z * f \quad (z \in \mathbb{C}^+;\ f \in L^2). \tag{7}$$

Since $k_z \in L^1$ $\big($with $\|k_z\|_1 = \big(\frac{|z|}{\Re z}\big)^{n/2}\big)$, $W(z)$ is well-defined (on *any* L^p), and

$$\|W(z)f\|_p \le \left(\frac{|z|}{\Re z}\right)^{n/2} \|f\|_p \quad (f \in L^p;\ z \in \mathbb{C}^+). \tag{8}$$

The absolute convergence of the convolution integral defining $[W(z)f](x)$ (for each fixed $x \in \mathbb{R}^n$) implies (by applying for example the Fubini and Morera theorems) that $W(\cdot)$ is an *analytic* $B(L^p)$-valued function extending $T(\cdot)$ to $\mathbb{C}^+$. The general L^p-estimate (8) is too weak to yield a "regularity" conclusion. Restricting the discussion as before to the case $p = 2$, we have by (6) and Parseval's formula (for $f \in \mathcal{S}$)

$$\|W(z)f\|_2 = \|k_z * f\|_2 = \|\mathcal{F}(k_z * f)\|_2 = \|\hat{k}_z\,(\mathcal{F}f)\|_2$$
$$= \|e^{-z|u|^2}(\mathcal{F}f)(u)\|_2 \le \|e^{-\Re z\,|u|^2}\|_\infty \|\mathcal{F}f\|_2 = \|f\|_2.$$

Since $\mathcal{S}$ is dense in L^2 and $W(z) \in B(L^2)$, it follows that $\|W(z)f\|_2 \le \|f\|_2$ for all $f \in L^2$. Thus $W(\cdot)$ *is an analytic contraction semigroup on* L^2*, which extends* $T(\cdot)$ *to* $\mathbb{C}^+$. In particular, $T(\cdot)$ *is regular, and its associated boundary group* $U(\cdot)$ *is a* C_o*-group of contractions, hence of isometries, hence a unitary group* on the Hilbert space L^2 (cf. Theorem 1.105). If A is the generator of $T(\cdot)$, then iA is the generator of the unitary group $U(\cdot)$, and therefore, by Stone's theorem, A *is selfadjoint.*

Let Δ denote the Laplacian

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

If $f \in \mathcal{S}$, then by (2)

$$(\mathcal{F}(t^{-1}[T(t)f - f])(u) = t^{-1}[e^{-t|u|^2} - 1](\mathcal{F}f)(u).$$

When $t \to 0+$, the right-hand side converges pointwise to $(-|u|^2)(\mathcal{F}f)(u)$, and is dominated by $|u|^2|(\mathcal{F}f)(u)| \in \mathcal{S} \subset L^2$. By dominated convergence, it converges to the above limit *in* L^2*-norm.* Applying the operator $\mathcal{F}^{-1} \in B(L^2)$, we conclude that

$$t^{-1}[T(t)f - f] \to \mathcal{F}^{-1}[-|u|^2(\mathcal{F}f)(u)] = \Delta f$$

in L^2*-norm.* This shows that $\mathcal{S} \subset D(A)$, and

$$Af = \Delta f \quad (f \in \mathcal{S}). \tag{9}$$

Since $\mathcal{S}$ is dense in L^2 and $T(\cdot)$-invariant (if $f \in \mathcal{S}$, then $T(t)f := k_t * f \in \mathcal{S}$ because $k_t \in \mathcal{S}$ and $\mathcal{S}$ is closed under convolution), it follows from Theorem 1.7 that $\mathcal{S}$ *is a core for* A.

The domain $D(A)$ is determined easily by using the L^2 *Fourier transform* F, that is, the unique extension of $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ as a unitary operator on L^2. Set

$$D_0 := \{f \in L^2;\ |u|^2(Ff)(u) \in L^2\}.$$

Let $f \in D_0$ and $t > 0$. Arguing as before, we have

$$F(t^{-1}[T(t)f - f])(u) = t^{-1}[e^{-t|u|^2} - 1](Ff)(u).$$

The right-hand side converges pointwise almost everywhere to $-|u|^2(Ff)(u)$, and is dominated by $|u|^2|(Ff)(u)| \in L^2$. By the Dominated Convergence Theorem, it converges in L^2 to its said pointwise limit. Since $F^{-1} \in B(L^2)$, it follows that

$$t^{-1}[T(t)f - f] \to F^{-1}[-|u|^2(Ff)(u)]$$

in L^2 norm. This shows that $D_0 \subset D(A)$ and Af is given by the last expression for $f \in D_0$. (Note that $\mathcal{S} \subset D_0$ trivially, and the expression obtained here for Af coincides with the former expression $Af = \Delta f$ for $f \in \mathcal{S}$.)

On the other hand, if $f \in D(A)$, then $t^{-1}[T(t)f - f] \to Af$ in L^2. Applying $F \in B(L^2)$, it follows that $g_t(u) := t^{-1}[e^{-t|u|^2} - 1](Ff)(u) \to F(Af)(u)$ in L^2. Therefore there exists a sequence $t_n \to 0+$ such that $g_{t_n} \to F(Af)$ pointwise almost everywhere. However, trivially, $g_{t_n}(u) \to -|u|^2(Ff)(u)$ (a.e.) as $n \to \infty$. Therefore $F(Af)(u) = -|u|^2(Ff)(u)$ a.e. In particular $|u|^2(Ff)(u) \in L^2$, hence $f \in D_0$, and we conclude that $D(A) = D_0$.

(By Theorem 1.2, the Gauss–Weierstrass semigroup $T(\cdot)$ and its associated boundary unitary group $U(\cdot)$ provide the solutions in L^2 of the Cauchy problem for the "heat equation" $\frac{\partial u}{\partial t} = \Delta u$, $t \geq 0$, and for the (free) "Schroedinger equation" $\frac{\partial v}{\partial t} = i\Delta v$, $t \in \mathbb{R}$, respectively.)

Example 1.115. The Cauchy–Poisson semigroup.

We consider now the *Cauchy–Poisson kernel*

$$k_t(x) = (\pi t)^{-1} \frac{1}{1 + (x/t)^2} \quad (t > 0;\, x \in \mathbb{R}). \tag{10}$$

Clearly $k_t \in L^1 := L^1(\mathbb{R})$, with L^1-norm $\|k_t\|_1 = 1$. Also

$$\hat{k}_t(u) = e^{-t|u|} \quad (t > 0;\, u \in \mathbb{R}) \tag{11}$$

(cf. [K17, Section I.3.14]).

The operator $T(t)$ defined on $L^p := L^p(\mathbb{R})$ by the convolution $T(t)f := k_t * f$ (for $t > 0$) is a *contraction* (since $\|k_t\|_1 = 1$).

By (11), if $s, t > 0$, $k_t * k_s$ has the Fourier transform $\hat{k}_t(u)\, \hat{k}_s(u) = e^{-(t+s)|u|} = \hat{k}_{t+s}(u)$, and it follows from the uniqueness property of the Fourier transform that $k_t * k_s = k_{t+s}$. Therefore, by associativity of the convolution (say, on $\mathcal{S} := \mathcal{S}(\mathbb{R})$), $T(t+s)f = T(t)T(s)f$ for all f in the dense subspace $\mathcal{S}$ of L^p (if $1 \leq p < \infty$). Since both $T(t)T(s)$ and $T(t+s)$ are in $B(L^p)$, we conclude that $T(\cdot)$, defined as above for $t > 0$ and defined as the identity at $t = 0$, is a *contraction semigroup on* L^p *for* $1 \leq p < \infty$.

Considering next a *complex* parameter $z \in \mathbb{C}^+$, the kernel

$$k_z(x) := (\pi z)^{-1} \frac{1}{1 + (x/z)^2} \quad (x \in \mathbb{R}) \tag{12}$$

is continuous on $\mathbb{R}$ (the denominator in (12) can vanish only if z is pure imaginary, which is contrary to our assumption that $z \in \mathbb{C}^+$), and is $O(1/x^2)$

for $|x| \to \infty$. Therefore $k_z \in L^1$, and consequently, the operators $W(z)$ defined on L^p by $W(z)f = k_z * f$ are bounded operators with norm $\leq \|k_z\|_1$. Also the *absolute convergence* of the Fourier integral defining $\hat{k}_z(u)$ (for each fixed $u \in \mathbb{R}$) implies that $\hat{k}_z(u)$ is an analytic function of z in $\mathbb{C}^+$. By (11), it agrees with the analytic function $e^{-z|u|}$ for $z \in (0, \infty)$. It follows that

$$\hat{k}_z(u) = e^{-z|u|} \quad (z \in \mathbb{C}^+;\ u \in \mathbb{R}). \tag{13}$$

Since $k_z \in L^q$ for any $q \geq 1$, the convolution integral $k_z * f$ converges absolutely for any $f \in L^p$ (where p, q are conjugate exponents and $z \in \mathbb{C}^+$), and since $k_z(x)$ is an analytic function of z in $\mathbb{C}^+$ for each fixed $x \in \mathbb{R}$, it follows that $W(\cdot)$ is an analytic $B(L^p)$-valued function in $\mathbb{C}^+$ $(1 \leq p < \infty)$), which extends the semigroup $T(\cdot)$.

As in Example 1.114, we shall restrict the following discussion to the case $p = 2$. We verify the C_o-property of $T(\cdot)$ by using (11) and Parseval's identity: for $t > 0$ and $f \in \mathcal{S}$

$$\|[T(t) - I]f\|_2 = (2\pi)^{-1/2}\|\hat{k}_t \hat{f} - \hat{f}\|_2 = \|(e^{-t|u|} - 1)(\mathcal{F}f)(u)\|_2.$$

The functions in the last L^2-norm sign are dominated by $2|\mathcal{F}f| \in L^2$ and converge to 0 as $t \to 0$; therefore their L^2-norms converge to 0. Since $\|T(t) - I\| \leq 2$ and $\mathcal{S}$ is dense in L^2, we conclude that $T(t) \to I$ in the s.o.t. on L^2, i.e., $T(\cdot)$ is a C_o-semigroup (of contractions) in L^2; this is the *Cauchy–Poisson semigroup* (in L^2).

We show now that the analytic extension $W(\cdot)$ of $T(\cdot)$ is *contraction-valued* on L^2. Indeed, for all $z \in \mathbb{C}^+$ and $f \in \mathcal{S}$, we have by (13) and Parseval's identity

$$\|W(z)f\|_2 = (2\pi)^{-1/2}\|\hat{k}_z \hat{f}\|_2 = \|e^{-z|u|}(\mathcal{F}f)(u)\|_2$$
$$\leq \sup_{u \in \mathbb{R}} e^{-\Re z\,|u|}\|\mathcal{F}f\|_2 = \|f\|_2.$$

Since $W(z) \in B(L^2)$, the inequality $\|W(z)f\|_2 \leq \|f\|_2$ extends to all $f \in L^2$ by the density of $\mathcal{S}$ in L^2.

In particular, $T(\cdot)$ *is a regular semigroup.*

Let A be its generator. Define

$$D_1 := \{f \in L^2;\ |u|(Ff)(u) \in L^2\},$$

where F denotes as before the L^2 Fourier transform. (Note that if $f \in D_0$ (cf. Example 1), then

$$\int_{\mathbb{R}} \||u|(Ff)(u)|^2\,du = (u^2(Ff)(u), (Ff)(u)) < \infty,$$

since Ff and $u^2(Ff)(u)$ are both in L^2. Thus $D_0 \subset D_1$.)

If $f \in D_1$, it makes sense to define

$$Bf := F^{-1}[-|u|(Ff)(u)]. \tag{14}$$

By (11),

$$F(t^{-1}[T(t) - I]f)(u) = t^{-1}[e^{-t|u|} - 1](Ff)(u). \tag{15}$$

The right-hand side of (15) converges pointwise to $-|u|(Ff)(u) = F(Bf)(u)$, and is dominated by $|u|\,|(Ff)(u)| \in L^2$; therefore, by the Dominated Convergence Theorem, it converges to $F(Bf)(u)$ in L^2-norm. Since $F^{-1} \in B(L^2)$, we conclude that $D_1 \subset D(A)$ and $Af = Bf$ for $f \in D_1$. Actually, we see that $D(A) = D_1$ by the argument we used in Example 1.114. (If $f \in D(A)$, since $F \in B(L^2)$, it follows from (15) that the right-hand side of (15) converges in L^2 to $F(Af)$ as $t \to 0+$; there exists therefore a sequence $t_n \to 0+$ for which the convergence is poinwise a.e. Since the pointwise a.e. limit is clearly $-|u|(Ff)(u)$, it follows that $|u|(Ff)(u) = -F(Af)(u) \in L^2$, i.e., $f \in D_1$, and we conclude that $D(A) = D_1$.)

As in Example 1.114, the generator A is selfadjoint, and *its square coincides with the generator of the Gauss–Weierstrass semigroup* (with equality of domains!).

Example 1.116 (The Gamma Semigroup).

In this example, we are concerned with convolution operators with the *Gamma kernel* $k_t(x)$ $\quad (t > 0;\ x \in \mathbb{R})$, where $k_t(x) = 0$ for $x \le 0$,

$$k_t(x) = \Gamma(t)^{-1} x^{t-1} e^{-bx} \tag{16}$$

for $x > 0$, and b is a positive constant.

One has (cf. [K17, Section 1.3.15])

$$\hat{k}_t(u) = (b - iu)^{-t} \quad (u \in \mathbb{R}). \tag{17}$$

We shall be concerned with the convolution operators $k_t * f$ with $f \in L^p := L^p(\mathbb{R}^+)$, where the later space is identified with the subspace of all $f \in L^p(\mathbb{R})$ vanishing (a.e.) on $(-\infty, 0]$. Clearly $k_t \in L^1$ with $\|k_t\|_1 = b^{-t}$. Therefore the operators

$$T(t) : f \in L^p \to k_t * f \quad (t > 0)$$

belong to $B(L^p)$ with operator norm $\le b^{-t}$. By (17), $\hat{k}_t\,\hat{k}_s = \hat{k}_{t+s}$, hence $k_t * k_s = k_{t+s}$ for $t, s > 0$ (by the uniqueness property of the L^1 *Fourier transform*), and therefore $T(t)T(s)f = T(t+s)f$ for all $f \in \mathcal{S}$ vanishing on $(-\infty, 0]$ (by the associative law for the convolution). Since $T(t)T(s)$ and $T(t+s)$ belong to $B(L^p)$, and the functions f as above are dense in L^p, it follows that $T(\cdot)$ is a semigroup of operators on L^p for any $p \in [1, \infty)$.

Using the same argument as in the former examples, one shows that $T(\cdot)$ can be extended as an analytic $B(L^p)$-valued function $W(\cdot)$ on $\mathbb{C}^+$, defined by

$$W(z)f = k_z * f \quad (f \in L^p;\ z \in \mathbb{C}^+), \tag{18}$$

where k_z is defined by (16) with t replaced by $z \in \mathbb{C}^+$. This follows from the fact that $k_z \in L^1$ for $z \in \mathbb{C}^+$, because

$$\|k_z\|_1 = \frac{\Gamma(\Re z)}{|\Gamma(z)|} b^{-\Re z} < \infty.$$

We may also apply the argument of the preceding examples to show that $\hat{k}_z$ is given by (17) with t replaced by z, that is,

$$\hat{k}_z(u) = (b - iu)^{-z} := (b^2 + u^2)^{-z/2} e^{-iz \arctan(u/b)} \quad (u \in \mathbb{R}). \tag{19}$$

In particular,

$$\|\hat{k}_z\|_\infty = b^{-\Re z} e^{(\pi/2)|\Im z|} \quad (z \in \mathbb{C}^+). \tag{20}$$

In case $p = 2$, the C_o-property follows as in the previous examples from the Parseval identity and (20) for $z \in (0, \infty)$:

$$\begin{aligned} \|T(t)f - f\|_2 &= \|F[k_t * f - f]\|_2 = \|(\hat{k}_t - 1)(Ff)\|_2 \\ &= \|[(b - iu)^{-t} - 1](Ff)(u)\|_2. \end{aligned}$$

The functions in the last norm sign converge pointwise a.e. to 0 when $t \to 0+$, and for $0 < t \le 1$, they are

$$\le (b^{-t} + 1)|Ff| \le K\,|Ff| \in L^2,$$

where $K = 1 + \max(1, 1/b)$. By the Dominated Convergence Theorem, they converge to 0 in L^2-norm, hence $T(t)f \to f$ in L^2 when $t \to 0+$.

For arbitrary $p \in [1, \infty)$, we may verify the C_o-property as follows. We first show that $T(1)$ *has dense range in* L^p. Indeed, let q be the conjugate exponent of p, and suppose $g \in L^q$ is such that $\langle T(1)f, g\rangle = 0$ for all $f \in L^p$. We must show that $g = 0$ a.e. We have

$$\begin{aligned} 0 &= \int_0^\infty \int_0^x e^{-b(x-y)} f(y)\, dy\, g(x)\, dx \\ &= \int_0^\infty \left(\int_y^\infty e^{-bx} g(x)\, dx \right) e^{by}\, f(y)\, dy \end{aligned}$$

for all $f \in L^p$, and therefore

$$e^{by} \int_y^\infty e^{-bx} g(x)\, dx = 0$$

a.e. on $\mathbb{R}^+$. Thus the above integral vanishes a.e., and therefore, for $h > 0$ small enough,

$$(1/2h) \int_{y-h}^{y+h} e^{-bx} g(x)\, dx = 0$$

for almost all $y \in \mathbb{R}^+$. Since the integrand clearly belongs to L^1, it follows that the left-hand side converges for almost all y (when $h \to 0+$) to $e^{-by}g(y)$. Consequently $g(y) = 0$ a.e.

Let $f \in L^p$ $(1 \leq p < \infty)$. Since $T(\cdot)f$ is the restriction of the *analytic* function $W(\cdot)f$ to $\mathbb{R}^+$, it is (strongly) continuous on $\mathbb{R}^+$. Therefore, as $t \to 0+$, $T(t)[T(1)f] = T(1+t)f \to T(1)f$, that is, $[T(t) - I]g \to 0$ strongly for all g in the range of $T(1)$. Since this range is dense in L^p and $\|T(t) - I\| \leq K$ for $0 < t \leq 1$ (K as above), it follows that $T(t)f \to f$ strongly for all $f \in L^p$. Thus, *for any* $p \in [1, \infty)$, $T(\cdot)$ *is a* C_o*-semigroup.*

Finally, we prove the regularity of $T(\cdot)$ (*for any* $1 < p < \infty$!) by applying the Mihlin Multiplier Theorem (cf. [K17, Corollary II.8.7]). Using the notation of this reference, we set

$$\|\mathcal{F}\, k_z\|_{\mathcal{K}} := \|\mathcal{F}\, k_z\|_\infty + \|M\, D\, \mathcal{F}\, k_z\|_\infty,$$

where $D : f \to f'$ and $M : f(u) \to uf(u)$ (whenever defined). We have

$$(M\, D\, \hat{k}_z)(u) = iuz(b - iu)^{-z-1},$$

hence

$$|(M\, D\, \hat{k}_z)(u)| = |z| \left(\frac{u^2}{b^2 + u^2}\right)^{1/2} (b^2 + u^2)^{-\Re z/2} e^{\Im z \arctan(u/b)}$$

$$\leq |z|\, (b^2 + u^2)^{-\Re z/2} e^{(\pi/2)|\Im z|}.$$

Hence

$$\|M\, D\, \hat{k}_z\|_\infty \leq |z|\, b^{-\Re z} e^{(\pi/2)|\Im z|}.$$

Together with (20), this gives

$$\|\mathcal{F}\, k_z\|_{\mathcal{K}} \leq (2\pi)^{-1/2} b^{-\Re z} (1 + |z|) e^{(\pi/2)|\Im z|}.$$

By Corollary II.8.7 in [K17], there exists a constant C_p depending only on p, $1 < p < \infty$, such that

$$\|W(z)\|_{B(L^p)} \leq C_p\, \|\mathcal{F}\, k_z\|_{\mathcal{K}} \leq (2\pi)^{-1/2} C_p b^{-\Re z} (1 + |z|) e^{(\pi/2)|\Im z|}.$$

Hence

$$\sup_{z \in Q} \|W(z)\|_{B(L^p)} \leq (2\pi)^{-1/2} C_p e^{\pi/2} (1 + \sqrt{2}) \max(1, 1/b) < \infty.$$

We conclude that, for any $p \in (1, \infty)$, *the Gamma semigroup* $T(\cdot)$ *is regular.* The associated boundary group $U(\cdot)$ can be shown to satisfy the estimate

$$\|U(t)\|_{B(L^p)} \leq C\, e^{(\pi/2)|t|} (1 + |t|),$$

uniformly with respect to the parameter b; the factor $C(1 + |t|)$ can be omitted in case $p = 2$ (cf. [K17, Section II.9.1, Relation (12)]).

In the above analysis of the Gamma semigroup, it was essential that the parameter b be positive. If $b = 0$, the operators $T(t)$, $t > 0$ (and $W(z)$, $z \in \mathbb{C}^+$) are *unbounded* on $L^p(\mathbb{R}^+)$; however, they may be considered on $L^p(0, N)$ for any $0 < N < \infty$, on which they are bounded operators; $W(\cdot)$ is an analytic semigroup on $L^p(0, N)$, the so-called *Riemann–Liouville semigroup* of "fractional integration":

$$[W(z)f](x) = \Gamma(z)^{-1} \int_0^x (x - y)^{z-1} f(y)\, dy$$

($f \in L^p(0, N)$; $z \in \mathbb{C}^+$; $x \in [0, N]$; $1 \leq p < \infty$). If $1 < p < \infty$, $W(\cdot)$ is *regular*; its boundary group $U(\cdot)$ is the group of "fractional integrals of pure imaginary order" (cf. [K17, Corollary II.9.3]).

G

Pre-Semigroups

We consider the following elementary properties of a C_o-semigroup $S(\cdot)$:

Property 1. $S(\cdot) : [0, \infty) \to B(X)$ is strongly continuous and $S(0)$ is injective.

Property 2. $S(t-u)S(u)$ is independent of u, for all $0 \leq u \leq t$.

Property 3. There exists $a \geq 0$ such that $e^{-at}S(t)x$ is bounded and uniformly continuous on $[0, \infty)$, for each $x \in X$.

Property 1 is contained in Theorem 1.1 (together with the trivial injectivity of $S(0) = I$). Property 2 follows from the semigroup identity. Property 3 follows from Theorem 1.1 and the estimate

$$\begin{aligned} &\|e^{-a(t+h)}S(t+h)x - e^{-at}S(t)x\| \\ &\qquad \leq e^{-at}\|S(t)\| \, \|e^{-ah}S(h)x - x\| \leq M\|e^{-ah}S(h)x - x\|. \end{aligned}$$

Definition 1.117. *A pre-semigroup is a function $S(\cdot)$ with the properties 1 and 2. If Property 3 is also satisfied, the pre-semigroup is said to be exponentially tamed.*

Let $S(\cdot)$ be a pre-semigroup. By Property 2, equating the values of $S(t-u)S(u)$ with the value at $u = t$, we see that

$$S(t-u)S(u) = S(0)S(t) \quad (t \geq u \geq 0). \tag{1}$$

Writing $t = u + s$ in (1), the identity is equivalent to

$$S(s)S(u) = S(0)S(u+s) \quad (s, u \geq 0). \tag{1'}$$

In particular, the values of $S(\cdot)$ commute.

S. Kantorovitz, *Topics in Operator Semigroups*,
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Definition 1.118. *The generator A of the pre-semigroup $S(\cdot)$ has domain $D(A)$ consisting of all $x \in X$ for which the strong right derivative at 0, $[S(\cdot)x]'(0)$, exists and belongs to $S(0)X$, and*

$$Ax := S(0)^{-1}[S(\cdot)x]'(0) \quad (x \in D(A)).$$

Note that if $T(\cdot)$ is a C_o-semigroup with generator A, and $C \in B(X)$ is injective and commutes with $T(\cdot)$, then $S(\cdot) := CT(\cdot)$ is a pre-semigroup with $S(0) = C$ and with generator A.

G.1 The Abstract Cauchy Problem

The generator A of a pre-semigroup is *not necessarily densely defined*, so that its associated ACP is more general than the one considered in Theorem 1.2. The following result is an adequate extension.

Theorem 1.119. *Let A generate the pre-semigroup $S(\cdot)$. Then:*

1. *A commutes with $S(t)$ for all $t \geq 0$.*
2. *A is closed with $S(0)X \subset \overline{D(A)}$.*
3. *For each $x \in D(A)$, $u := S(\cdot)x$ is of class C^1 and solves the Abstract Cauchy Problem*
$$u' = Au; \quad u(0) = S(0)x \tag{ACP}$$
on $[0, \infty)$.

Proof. For $t \geq 0$, $h > 0$, and $x \in D(A)$,

$$S(h)[S(t)x] - S(0)[S(t)x] = S(t)[S(h)x - S(0)x] = S(0)S(t+h)x - S(0)S(t)x.$$

Dividing by h and letting $h \to 0$, we get that the strong right derivative at 0 of $S(\cdot)[S(t)x]$ exists, equals the strong right derivative of $S(0)S(\cdot)x$ at t, and equals $S(t)S(0)Ax = S(0)S(t)Ax \in S(0)X$. Therefore $S(t)x \in D(A)$ and $A[S(t)x] := S(0)^{-1}[S(0)S(t)Ax] = S(t)Ax$. This proves Statement 1.

Also for $0 < h \leq t$ (with t fixed), letting $K := \sup_{0<u\leq t} \|S(u)\| (< \infty$ by the Uniform Boundedness Theorem and Property 1), we have

$$\begin{aligned}
&\|h^{-1}[S(0)S(t-h)x - S(0)S(t)x] + S(0)S(t)Ax\| \\
&\quad \leq \|h^{-1}[S(0)S(t-h) - S(0)S(t)]S(h)x + S(0)S(t)Ax\| \\
&\quad + \|S(0)\| \, \|S(t-h) - S(t)\| \, \|h^{-1}[S(h)x - x] - S(0)Ax\| \\
&\quad + \|S(0)\| \, \|S(t)[S(0)Ax] - S(t-h)[S(0)Ax]\|.
\end{aligned}$$

The first term on the right of the inequality equals

$$\|S(0)\{-h^{-1}[S(0)S(t+h) - S(0)S(t)] + S(t)Ax\}\| \to 0$$

when $h \to 0$, as observed before.

The second term on the right is

$$\leq 2K \,\|S(0)\|\, \|h^{-1}[S(h)x - x] - S(0)Ax\| \to 0$$

by definition of A.

The third term on the right $\to 0$ by continuity of $S(\cdot)[S(0)Ax]$.

Thus we proved that $S(0)S(\cdot)x$ is differentiable on $[0,\infty)$ for each $x \in D(A)$, and

$$[S(0)S(\cdot)x]'(t) = S(0)S(t)Ax \quad (t \geq 0; x \in D(A)). \tag{2}$$

Integrating from 0 to t and using the boundedness and injectivity of $S(0)$, we obtain

$$S(t)x - S(0)x = \int_0^t S(s)Ax\,ds = \int_0^t AS(s)x\,ds \quad (x \in D(A)). \tag{3}$$

For $h, t > 0$ and for all $x \in X$, we have

$$\begin{aligned} h^{-1}[S(h) - S(0)] \int_0^t S(s)x\,ds &= S(0)h^{-1}\left[\int_0^t S(s+h)x\,ds - \int_0^t S(s)x\,ds\right] \\ &= S(0)\left[h^{-1}\int_t^{t+h} S(s)x\,ds - h^{-1}\int_0^h S(s)x\,ds\right] \\ &\to S(0)[S(t)x - S(0)x] \in S(0)X, \end{aligned}$$

showing that $\int_0^t S(s)x\,ds \in D(A)$ and

$$A\int_0^t S(s)x\,ds = S(t)x - S(0)x \quad (x \in X). \tag{4}$$

In particular, for all $x \in X$,

$$S(0)x = \lim_{t\to 0+} t^{-1}\int_0^t S(s)x\,ds \in \overline{D(A)}.$$

We show now that A is closed. If $x_n \in D(A)$, $x_n \to x$, and $Ax_n \to y$, then with K as before (for t fixed) and $L = \sup_n \|Ax_n\|$, we have $\|S(s)Ax_n\| \leq KL$ for all n and $s \in [0,t]$, and $S(s)Ax_n \to S(s)y$ pointwise. By dominated convergence and (3),

$$S(t)x - S(0)x = \lim_n [S(t)x_n - S(0)x_n] = \lim_n \int_0^t S(s)Ax_n\,ds = \int_0^t S(s)y\,ds.$$

Dividing by t and letting $t \to 0+$, we obtain that the right-hand side converges to $S(0)y \in S(0)X$, so that $x \in D(A)$ and $Ax = y$, as wanted.

We read also from (3) that $S(\cdot)x$ is of class C^1 and solves (ACP) on $[0,\infty)$ for each $x \in D(A)$. □

A partial converse of Theorem 1.119 is the following

Theorem 1.120. *Let $S(\cdot)$ have Property 1 and commute with A, and either $D(A)$ is dense or $\rho(A)$ is nonempty. If $S(\cdot)x$ solves (ACP) for each $x \in D(A)$, then $S(\cdot)$ is a pre-semigroup generated by an extension of A.*

Proof. For $x \in D(A)$ and $0 \le u \le t$,

$$\frac{d}{du} S(t-u)S(u)x = -AS(t-u)S(u)x + S(t-u)AS(u)x = 0,$$

and Property 2 follows on $D(A)$, hence on X in case $D(A)$ is dense, because $S(t-u)S(u) \in B(X)$.

In case $\rho(A)$ is nonempty, fix $\lambda \in \rho(A)$. Since $R(\lambda; A)x \in D(A)$ for all $x \in X$, and since $R(\lambda; A)$ commutes with $S(\cdot)$ (because A commutes with $S(\cdot)$), we have

$$\begin{aligned} R(\lambda; A)S(t-u)S(u)x &= S(t-u)S(u)R(\lambda; A)x \\ &= S(0)S(t)R(\lambda; A)x = R(\lambda; A)S(0)S(t)x, \end{aligned}$$

and therefore $S(t-u)S(u) = S(0)S(t)$, i.e., Property 2 is satisfied.

Let then A' be the generator of $S(\cdot)$. By hypothesis, $[S(\cdot)x]'(0) = A[S(0)x] = S(0)Ax \in S(0)X$ for all $x \in D(A)$, that is, $A \subset A'$. □

A generalization of Theorem 1.120 is the following

Theorem 1.121. *Let A, B be (unbounded) operators such that*

(i) $0 \in \rho(B)$;
(ii) $D(B) \subset D(A)$; *and*
(iii) B *commutes with* $R(\lambda; A)$ *for some* $\lambda > 0$.

Then (ACP) for A has a unique C^1-solution on $[0, \infty)$ for each $x \in D(B)$ if and only if an extension of A generates a pre-semigroup $S(\cdot)$ (with $S(0) = (\lambda I - A)R(0; B)$) that commutes with A.

Proof. Suppose $S(\cdot)$ is a pre-semigroup with $S(0)$ as stated, commuting with A and generated by an extension A' of A. First, $S(t)D(A) = S(t)R(\lambda; A)X = R(\lambda; A)S(t)X \subset D(A)$ for all t. For $x = S(0)y$ with $y \in D(A) \subset D(A')$, we have by Theorem 1.119

$$[S(\cdot)y]' = A'[S(\cdot)y] = A[S(\cdot)y],$$

i.e., $[S(\cdot)S(0)^{-1}x]' = A[S(\cdot)S(0)^{-1}x]$ and of course $[S(\cdot)S(0)^{-1}x](0) = x$, that is, $S(\cdot)S(0)^{-1}x$ solves (ACP) for $x \in S(0)D(A) = D(B)$, since

$$S(0)D(A) = S(0)R(\lambda; A)X = R(\lambda; A)S(0)X = R(0; B)X = D(B).$$

If $v : [0,\infty) \to D(A)$ is any solution of (ACP) with $x = S(0)y \in S(0)D(A)$, then $\frac{d}{ds}[S(t-s)v(s)] = -S(t-s)A'v(s)+S(t-s)Av(s) = 0$ since $v(s) \in D(A)$. Equating therefore the values of the constant function $s \to S(t-s)v(s)$ at $s = 0$ and $s = t$, we get $S(0)v(t) = S(t)S(0)y$, hence $v(t) = S(t)y = S(t)S(0)^{-1}x$, meaning that (ACP) has a unique solution for each $x \in S(0)D(A)$.

Conversely, assume (ACP) with initial value $x \in D(B)$ has the unique C^1-solution $u(\cdot;x)$ on $[0,\infty)$. If $v := R(\lambda;A)u(\cdot;x)$, then

$$v' = R(\lambda;A)u(\cdot;x)' = R(\lambda;A)Au(\cdot;x) = Av,$$

and

$$v(0) = R(\lambda;A)x = R(\lambda;A)R(0;B)y = R(0;B)R(\lambda;A)y \in D(B).$$

By the uniqueness assumption,

$$R(\lambda;A)u(\cdot;x) = u(\cdot;R(\lambda;A)x) \quad (x \in D(B)). \tag{5}$$

We define now for all $x \in X$

$$S(\cdot)x := (\lambda I - A)u(\cdot;R(0;B)x) = \lambda u(\cdot;R(0;B)x) - u'(\cdot;R(0;B)x). \tag{6}$$

Since $R(0;B)x \in D(B)$, and $u(\cdot;y)$ has values in $D(A)$ for any $y \in D(B)$, the operator $S(t)$ is everywhere defined on X, and is linear by the uniqueness hypothesis (for each $t \geq 0$). By (6), $S(\cdot)x$ is continuous for each $x \in X$.

By (5) with $R(0;B)x (\in D(B))$ replacing x,

$$\begin{aligned}
R(\lambda;A)S(t)x &= u(t;R(0;B)x) = (\lambda I - A)R(\lambda;A)u(t;R(0;B)x)\\
&= (\lambda I - A)u(t;R(\lambda;A)R(0;B)x)\\
&= (\lambda I - A)u(t;R(0;B)R(\lambda;A)x)\\
&= S(t)R(\lambda;A)x,
\end{aligned}$$

and it follows that $S(t)$ commutes with A for all t.

Consider now $U(\cdot)x := u(\cdot;R(0;B)x)$.

The operator $U(\cdot) : X \to C^1([0,b];X)$ (:= the Banach space of all X-valued C^1-functions on $[0,b]$ with the usual norm) is shown to be closed. Indeed, if $x_n \to x$ in X, and $U(\cdot)x_n \to v$ in $C^1([0,b];X)$, then for each $t \in [0,b]$,

$$\begin{aligned}
A[U(t)x_n] &= Au(t;R(0;B)x_n) = [u(\cdot;R(0;B)x_n)]'(t)\\
&= [U(\cdot)x_n]'(t) \to v'(t).
\end{aligned}$$

Since A is closed, $v(t) \in D(A)$ and $Av(t) = v'(t)$. Also

$$v(0) = \lim_n U(0)x_n = \lim_n u(0; R(0;B)x_n)$$
$$= \lim_n R(0;B)x_n = R(0;B)x.$$

By uniqueness, it follows that $v = u(\cdot; R(0;B)x) = U(\cdot)x$, that is, $U(\cdot)$ is closed, hence bounded, by the Closed Graph Theorem. Let M denote its norm. Then for $0 \le t \le b$,

$$\|S(t)x\| = \|(\lambda I - A)U(t)x\| \le \lambda\|U(t)x\| + \|[U(\cdot)x]'(t)\|$$
$$\le (\lambda + 1)M\|x\| \quad (x \in X).$$

Since b is arbitrary, this shows that $S(\cdot)$ is $B(X)$-valued. We saw that it satisfies Property 1 with $S(0) = (\lambda I - A)R(0;B)x$ clearly injective, and it commutes with A. For $x \in D(A)$, write $x = R(\lambda; A)y$; then by (5),

$$S(\cdot)x = (\lambda I - A)u(.; R(0;B)R(\lambda;A)y) = u(\cdot; R(0;B)y)$$

solves (ACP) (with the initial value $S(0)x = R(0;B)y$).

By Theorem 1.120, we conclude that $S(\cdot)$ is a pre-semigroup generated by an extension of A. □

Taking $B = -(\lambda I - A)^{n+1}$ (for some nonnegative integer n) with domain $D(B) = D(A^{n+1}) \subset D(A)$, the pre-semigroup $S(\cdot)$ generated by an extension of A satisfies $S(0) = (\lambda I - A)(\lambda I - A)^{-n-1} = R(\lambda; A)^n$. We thus have the following

Corollary 1.122. *Let $\lambda \in \rho(A)$. Then (ACP) for A has a unique C^1-solution on $[0,\infty)$ for each $x \in D(A^{n+1})$ if and only if an extension of A generates a pre-semigroup $S(\cdot)$ with $S(0) = R(\lambda; A)^n$, which commutes with A.*

G.2 The Exponentially Tamed Case

We consider next an *exponentially tamed* pre-semigroup $S(\cdot)$, that is, all three properties 1, 2, 3 are satisfied by $S(\cdot)$.

By the injectivity of $S(0)$ and the Uniform Boundedness Theorem,

$$0 < \|S(0)\| \le M := \sup_{t \ge 0} e^{-at}\|S(t)\| < \infty.$$

Definition 1.123. *Let $S(\cdot)$ be a pre-semigroup with Property 3. Set*

$$Y = \{x \in X; S(0)^{-1}e^{-at}S(t)x \in C_b([0,\infty); X)\},$$

where $C_b(\cdots)$ denotes the Banach space of all X-valued bounded uniformly continuous functions on $[0,\infty)$, normed by $\|f\|_u = \sup_{t\ge 0}\|f(t)\|$.

For $x \in Y$, set

$$\|x\|_Y := \|S(0)^{-1}e^{-at}S(t)x\|_u.$$

Clearly $\|\cdot\|_Y \geq \|\cdot\|$ on Y, and $Y := (Y, \|\cdot\|_Y)$ is a normed space.

We denote by $[S(0)X]$ the Banach space $S(0)X$ with the norm

$$\|x\|_0 := M\|S(0)^{-1}x\|.$$

The next result identifies Y as a Banach subspace of the Hille–Yosida space Z for the generator A of the exponentially tamed pre-semigroup $S(\cdot)$ (cf. Theorem 1.23).

Theorem 1.124. *Let A generate an exponentially tamed pre-semigroup $S(\cdot)$, and let Y be the space defined in Definition 1.123. Then Y is a Banach subspace of X containing $[S(0)X]$ as a Banach subspace, and A_Y (the part of A in Y) generates a C_o-semigroup $T(\cdot)$ in Y satisfying $\|T(t)\|_{B(Y)} \leq e^{at}$.*

Proof. Let $\{x_n\}$ be Cauchy in Y. Since $\|\cdot\|_Y \geq \|\cdot\|$, it is Cauchy in X. Let then $x = \lim x_n$ in X. By definition of the Y-norm, the sequence $\{S(0)^{-1}e^{-at}S(t)x_n\}$ is Cauchy in $C_b := C_b([0,\infty);X)$. Let $u \in C_b$ be its C_b-limit. The boundedness of $S(0)$ implies that $e^{-at}S(t)x = S(0)u(t) \in S(0)X$ for all t, so that $S(0)^{-1}e^{-at}S(t)x = u(t) \in C_b$, i.e., $x \in Y$, and clearly $\|x_n - x\|_Y \to 0$. Thus Y is indeed a Banach subspace of X.

If $x = S(0)y \in S(0)X$, then $S(0)^{-1}e^{-at}S(t)x = e^{-at}S(t)y \in C_b$ by Property 3, so that $S(0)X \subset Y$. Also

$$\|x\|_Y \leq \|e^{-at}S(t)y\|_u \leq M\,\|y\| = \|x\|_0.$$

Hence $[S(0)X]$ is a Banach subspace of Y.

If $x \in Y$, then for each fixed $s \geq 0$, $S(0)^{-1}e^{-at}S(t)S(s)x = e^{-at}S(t+s)x \in C_b$ (as a function of t), so that $S(s)Y \subset Y$. We may then define $T(\cdot) := S(0)^{-1}S(\cdot)$ on Y. Then for $x \in Y$,

$$\|T(t)x\|_Y = \sup_s \|S(0)^{-1}e^{-as}S(s)S(0)^{-1}S(t)x\|$$

$$= e^{at}\sup_s \|S(0)^{-1}e^{-a(s+t)}S(s+t)x\| \leq e^{at}\|x\|_Y.$$

This shows that $T(\cdot)$ is $B(Y)$-valued and $\|T(t)\|_{B(Y)} \leq e^{at}$.

The semigroup property of $T(\cdot)$ follows trivially from $(1')$.

We shall verify the C_o-property of $T(\cdot)$ by using the uniform continuity of $S(0)^{-1}e^{-at}S(t)x$ for $x \in Y$. Given $\epsilon > 0$, there exists $\delta > 0$ such that for $0 < h < \delta$,

$$\|S(0)^{-1}e^{-a(t+h)}S(t+h)x - S(0)^{-1}e^{-at}S(t)x\|_u < \epsilon.$$

Therefore

$$\|S(0)^{-1}e^{-at}S(t)[T(h)x - x]\|$$
$$\leq e^{ah}\|S(0)^{-1}e^{-a(t+h)}S(t+h)x - S(0)^{-1}e^{-at}S(t)x\|$$
$$+ (e^{ah} - 1)\|S(0)^{-1}e^{-at}S(t)x\| \leq e^{ah}\epsilon + (e^{ah} - 1)\|x\|_Y.$$

Taking the supremum over all $t \geq 0$, and letting then $h \to 0$, we obtain

$$\limsup_{h \to 0+} \|T(h)x - x\|_Y \leq \epsilon,$$

and the arbitrariness of ϵ gives the C_o-property of $T(\cdot)$ in Y.

Let then A' be the generator of $T(\cdot)$ in Y. For $x \in D(A') \subset Y$ and $h > 0$,

$$h^{-1}[S(h)x - S(0)x] = S(0)h^{-1}[T(h)x - x] \to S(0)A'x \in S(0)X,$$

where the limit (as $h \to 0$) is in Y, hence in X. Therefore $x \in D(A)$ and $Ax = A'x \in Y$, by definition. This shows that $x \in D(A_Y)$ and $A_Y x = A'x$, i.e., $A' \subset A_Y$.

The Laplace transform $L(\lambda)x$ of $S(\cdot)x$ is well-defined for $\lambda > a$, and calculations identical with those in the proof of Theorem 1.15 show that

$$L(\lambda)(\lambda I - A)x = S(0)x \quad (x \in D(A)).$$

If $(\lambda I - A)x = 0$, it follows that $S(0)x = 0$, hence $x = 0$, i.e., $(\lambda I - A)$ is injective on $D(A)$, and therefore $(\lambda I - A_Y)$ is injective (for $\lambda > a$). Also, by Theorem 1.15, $R(\lambda; A') \in B(Y)$, so that in particular $\lambda I - A'$ is surjective (for those λ), and we saw above that $\lambda I - A' \subset \lambda I - A_Y$. It follows that $D(A') = D(A_Y)$. □

Part II

Integral Representations

A

The Semi-Simplicity Space

Motivated by the spectral integral representation of bounded C_o-groups of operators in Hilbert space (cf. Corollary 1.42) and its Banach space version on the semi-simplicity space (cf. Theorem 1.49 and Corollary 1.50), we shall construct the semi-simplicity space for operators A for which iA *is not necessarily the generator of a* C_o*-group*, provided that the spectrum of A lies on the real line, or at least excludes some ray. We shall then prove the existence of a spectral integral representation of the part of A in its semi-simplicity space, and the maximality of the latter with respect to this property.

A.1 The Real Spectrum Case

In this section, we generalize Theorem 1.49 to operators A *with real spectrum*, for which iA *is not assumed to generate a* C_o*-group*.

Consider the *Poissonian* of A,

$$P(t,s) := \frac{1}{2\pi i}[R(t-is;A) - R(t+is;A)] \quad (t \in \mathbb{R}; s > 0).$$

Let $\|\cdot\|_1$ denote the $L^1(\mathbb{R})$-norm (with respect to the Lebesgue measure).

Definition 2.1. *The semi-simplicity space for A is the set of all $x \in X$ such that*

1. $\lim_{|u|\to\infty} R(t+iu;A)x = 0$ *for all* $t \in \mathbb{R}$*; and*
2. $\sup_{s>0} \|x^* P(\cdot,s)x\|_1 < \infty$ *for all* $x^* \in X^*$.

Note that Condition 1 is valid for all $x \in X$ when iA generates a C_o-group; indeed, since $R(t+iu;A) = iR(-u+it;iA)$, we have in that case $\|R(t+iu;A)\| = O(1/|u|)$ for $0 \neq u \in \mathbb{R}$.

Lemma 2.2. *If $x \in X$ satisfies Condition 2, then*

$$\|x\|_A := \sup\{\|x^* P(.,s)x\|_1; s > 0, \|x^*\| = 1\} < \infty.$$

S. Kantorovitz, *Topics in Operator Semigroups*,
Progress in Mathematics 281, DOI 10.1007/978-0-8176-4932-6_8,

Proof. First, for $s > 0$ and $x \in X$ fixed, *assume* that $x^* P(.,s)x \in L^1(\mathbb{R})$ for all $x^* \in X^*$, and consider then the linear map

$$V_s : x^* \to x^* P(\cdot, s)x$$

of X^* into $L^1(\mathbb{R})$.

If $x_n^* \to x^*$ in X^* and $V_s x_n^* \to f$ in L^1 (as $n \to \infty$), then by Fatou's lemma,

$$\int_{\mathbb{R}} |f(t) - x^* P(t,s)x|\, dt = \int_{\mathbb{R}} \liminf_n |f(t) - x_n^* P(t,s)x|\, dt$$

$$\leq \liminf_n \int |f(t) - V_s x_n^*|\, dt = 0,$$

i.e., $V_s x^* = f$, so that V_s is closed, hence bounded, by the Closed Graph Theorem.

When Condition 2 is satisfied by x, the family of *bounded* operators $\{V_s;\ s > 0\}$ satisfies

$$\sup_{s>0} \|V_s x^*\|_1 < \infty \quad (x^* \in X^*).$$

By the Uniform Boundedness Theorem, $\sup_{s>0} \|V_s\| < \infty$, which means precisely that $\|x\|_A < \infty$. □

Theorem 2.3. *Let A be an operator with real spectrum acting in the reflexive Banach space X, and let Z be its semi-simplicity space, normed by $\|\cdot\|_A$. Then Z is a Banach subspace of X, invariant for any $U \in B(X)$ commuting with A, and there exists a spectral measure on Z, $E(\cdot)$, such that*

1. *for each $\delta \in \mathcal{B}(\mathbb{R})$, $E(\delta)$ commutes with every $U \in B(X)$ which commutes with A;*
2. *$D(A_Z) = \{x \in Z; \int_{\mathbb{R}} uE(du)x$ exists and belongs to $Z\}$, and*

$$Ax = \int_{\mathbb{R}} uE(du)x \quad (x \in D(A_Z));$$

3. *for all nonreal $\zeta \in \mathbb{C}$ and $x \in Z$,*

$$R(\zeta; A)x = \int_{\mathbb{R}} \frac{1}{\zeta - u} E(du)x.$$

Moreover, Z is "maximal-unique" in the following sense: if W is a Banach subspace of X and $F(\cdot)$ is a spectral measure on W with Property 3, then $W \subset Z$ and $F(\delta) = E(\delta)|_W$ for all $\delta \in \mathcal{B}(\mathbb{R})$.

Note that the "existence" of the integral in Statement 2 is in the sense of Section 1.48, i.e., as the strong limit in X

$$\int_{\mathbb{R}} uE(du)x := \lim_{a,b} \int_a^b uE(du)x.$$

Proof. For fixed $x \in Z$ and $x^* \in X^*$, the function $x^*P(t,s)x$ is an analytic function of $t+is$ in $\mathbb{C}^+$, hence (complex) harmonic there, and satisfies

$$\sup_{s>0} \|x^*P(\cdot,s)x\|_1 \le \|x\|_A \|x^*\| < \infty$$

(by Lemma 2.2).

Therefore there exists a unique regular complex Borel measure $\mu(\cdot;x,x^*)$ on $\mathcal{B}(\mathbb{R})$ such that

$$\|\mu(\cdot;x,x^*)\| \le \|x\|_A \|x^*\| \tag{1}$$

and

$$x^*P(t,s)x = \int_{\mathbb{R}} p(t-u,s)\mu(du;x,x^*) \quad (t \in \mathbb{R}, s>0), \tag{2}$$

where $p(t,s) := \frac{s}{\pi(t^2+s^2)}$ is the Poisson kernel for the upper halfplane (cf. [SW], pp. 49–53). The uniqueness of the Poisson integral representation implies that for each $\delta \in \mathcal{B}(\mathbb{R})$, $\mu(\delta;\cdot,\cdot)$ is a bilinear form; hence, by (1) and the reflexivity of X, there exists a unique linear transformation

$$E(\delta) : Z \to X$$

such that

$$\mu(\delta;x,x^*) = x^*E(\delta)x \quad (x \in Z,\, x^* \in X^*) \tag{3}$$

and

$$\|E(\delta)x\| \le \|x\|_A \quad (x \in Z). \tag{4}$$

It follows from (3) and Pettis' theorem that $E(\cdot)x$ is a regular countably additive X-valued measure on $\mathcal{B}(\mathbb{R})$, and we may rewrite (2) in the form

$$[R(t-is;A) - R(t+is;A)]x = \int_{\mathbb{R}} \left[\frac{1}{t-is-u} - \frac{1}{t+is-u}\right] E(du)x \tag{5}$$

for all $t \in \mathbb{R}, s>0$, and $x \in Z$.

For $x \in Z$ fixed, consider the function

$$F(\zeta)x := \int_{\mathbb{R}} \frac{1}{\zeta - u} E(du)x \quad (\zeta \in \mathbb{C} - \mathbb{R}).$$

By (4), $F(\cdot)x$ is well-defined, analytic in $\mathbb{C}-\mathbb{R}$, and it follows from (1) that

$$\|F(\zeta)x\| \le \frac{\|x\|_A}{|\Im \zeta|}.$$

In particular, $F(t+is)x \to 0$ as $|s| \to \infty$. By (5),

$$F(t-is)x - F(t+is)x = R(t-is;A)x - R(t+is;A)x \tag{6}$$

for all $t \in \mathbb{R}, s>0$, and $x \in Z$.

Set $G(\cdot)x = F(\cdot)x - R(\cdot; A)x$. This is an analytic function in $\mathbb{C} - \mathbb{R}$, satisfying $G(\zeta)x = \overline{G(\overline{\zeta})x}$ in its domain. Therefore, for each $x^* \in X^*$, the functions $x^*G(\zeta)x$ and $\overline{x^*G(\zeta)x} (= \overline{x^*G(\overline{\zeta})x})$ are both analytic in $\mathbb{C} - \mathbb{R}$. Hence $x^*G(\cdot)x$ is constant there, and since it vanishes as $|\Im\zeta| \to \infty$ (by Condition 1 in Definition 2.1, and by our previous observation about $F(\cdot)x$), it follows that $G(\cdot)x = 0$ for all $x \in Z$, i.e.,

$$R(\zeta; A)x = \int_{\mathbb{R}} \frac{1}{\zeta - u} E(du)x \tag{7}$$

for all $\zeta \in \mathbb{C} - \mathbb{R}$ and $x \in Z$.

We shall verify now that (7) (i.e., Statement 3 of our theorem) implies all the other statements of Theorem 2.3.

Let $U \in B(X)$ commute with A. Then U commutes with $R(\zeta; A)$ for all nonreal ζ. If $x \in Z$, then $R(t + iu; A)Ux = UR(t + iu; A)x \to 0$ as $|u| \to \infty$, for all $t \in \mathbb{R}$. Also, with notations as in the proof of Lemma 2.2, we have for fixed $x \in Z$,

$$\begin{aligned}
\|x^*P(\cdot, s)Ux\|_1 &= \|x^*UP(\cdot, s)x\|_1 = \|(U^*x^*)P(\cdot, s)x\|_1 \\
&= \|V_sU^*x^*\|_1 \le \|V_s\| \, \|U^*x^*\| \le \|V_s\| \, \|U\| \, \|x^*\| \\
&\le \|U\| \, \|x\|_A \, \|x^*\|,
\end{aligned}$$

so that

$$\|Ux\|_A \le \|U\| \, \|x\|_A \quad (x \in Z). \tag{8}$$

Thus Z is U-invariant, and by (7),

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{\zeta - u} E(du)Ux &= R(\zeta; A)Ux = UR(\zeta; A)x \\
&= \int_{\mathbb{R}} \frac{1}{\zeta - u} UE(du)x
\end{aligned}$$

for all nonreal ζ and $x \in Z$.

By the uniqueness property of the Stieltjes transform (cf. [W]), it follows that $E(\delta)Ux = UE(\delta)x$ for all $x \in Z$ and $\delta \in \mathcal{B}(\mathbb{R})$ (which proves Statement 1 of the theorem).

In particular, taking $U = R(\lambda; A)$ for $\lambda \in \rho(A)$, we obtain

$$\begin{aligned}
R(\lambda; A)E(\mathbb{R})x &= E(\mathbb{R})R(\lambda; A)x \\
&= \lim_{\Im\zeta \to \infty} \int_{\mathbb{R}} \frac{\zeta}{\zeta - u} E(du)R(\lambda; A)x = \lim \zeta R(\zeta; A)R(\lambda; A)x \\
&= \lim \frac{\zeta}{\zeta - \lambda}[R(\lambda; A) - R(\zeta; A)]x = R(\lambda; A)x,
\end{aligned}$$

since $x \in Z$ (cf. Condition 1 in Definition 2.1). Hence

$$E(\mathbb{R})x = x \quad (x \in Z). \tag{9}$$

By (4), this shows in particular that

$$\|x\| \le \|x\|_A \quad (x \in Z). \tag{10}$$

Therefore, if $\{x_n\} \subset Z$ is $\|\cdot\|_A$-Cauchy, it is also $\|\cdot\|$-Cauchy (and of course, $\|\cdot\|_A$-bounded, say by the constant K). Let $x = \lim_n x_n$ in X. Then by (7) and (1),

$$\|R(t+iu;A)x_n\| \le \frac{\|x_n\|_A}{|u|} \le \frac{K}{|u|},$$

and therefore $\|R(t+iu;A)x\| \le \frac{K}{|u|}$. Thus x satisfies Condition 1 in Definition 2.1.

For each $x^* \in X^*$ and $s > 0$, $x^*P(\cdot,s)x_n \to_n x^*P(\cdot,s)x$ pointwise, so that by Fatou's lemma,

$$\begin{aligned}\|x^*P(\cdot,s)x\|_1 &\le \liminf_n \|x^*P(\cdot,s)x_n\|_1 \\ &\le \liminf_n \|x_n\|_A\,\|x^*\| \le K\|x^*\|,\end{aligned}$$

and we conclude that $x \in Z$.

Also, given $\epsilon > 0$, let n_o be such that $\|x_n - x_m\|_A < \epsilon$ for all $n, m > n_o$. Then $\|x^*P(\cdot,s)(x_n - x_m)\|_1 < \epsilon$ for all unit vectors $x^* \in X^*, s > 0$, and $n, m > n_o$. Letting $m \to \infty$, Fatou's lemma implies that $\|x_n - x\|_A \le \epsilon$ for all $n > n_o$, i.e., $x_n \to x$ in the $\|\cdot\|_A$-norm. We then conclude that $(Z, \|\cdot\|_A)$ is a Banach subspace of X.

For $x \in Z$ and $\lambda \in \rho(A)$, we have by (7), the invariance of Z under $R(\lambda;A)$, and the Resolvent Identity

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{\zeta - u} E(du)R(\lambda;A)x &= R(\zeta;A)R(\lambda;A)x \\ &= \frac{1}{\zeta-\lambda}[R(\lambda;A) - R(\zeta;A)]x \\ &= \frac{1}{\zeta-\lambda}\int_{\mathbb{R}}\left[\frac{1}{\lambda-u} - \frac{1}{\zeta-u}\right] E(du)x \\ &= \int_{\mathbb{R}} \frac{1}{\zeta-u}\left[\frac{1}{\lambda-u}E(du)x\right],\end{aligned}$$

for all nonreal ζ. By the uniqueness property of the Stieltjes transform,

$$E(\delta)R(\lambda;A)x = \int_{\mathbb{R}} \frac{1}{\lambda-u}\chi_\delta(u)E(du)x, \tag{11}$$

for all $x \in Z, \lambda \in \rho(A)$, and $\delta \in \mathcal{B}(\mathbb{R})$ (where χ_δ denotes the characteristic function of δ).

Since $R(\lambda; A)$ commutes with $E(\delta)$, it follows from (11) that for $x \in Z$,

$$R(\lambda; A)E(\delta)x = \int_\delta \frac{1}{\lambda - u} E(du)x \to 0$$

when $|\Im\lambda| \to \infty$.

Also for all unit vectors $x^* \in X^*$, we have by (11) and (1)

$$\begin{aligned} \|x^* P(\cdot, s)E(\delta)x\|_1 &= \|x^* E(\delta)P(\cdot, s)x\|_1 \\ &= \frac{s}{\pi} \int_{\mathbb{R}} \left| x^* \int_\delta \frac{1}{(t-u)^2 + s^2} E(du)x \right| dt \\ &\le \int_\delta \frac{s}{\pi} \int_{\mathbb{R}} \frac{dt}{(t-u)^2 + s^2} |x^* Ex|(du) \\ &= |x^* Ex|(\delta) \le \|\mu(\cdot; x, x^*)\| \le \|x\|_A. \end{aligned}$$

Hence $E(\delta)Z \subset Z$ and

$$\|E(\delta)x\|_A \le \|x\|_A \quad (x \in Z, \delta \in \mathcal{B}(\mathbb{R})). \tag{12}$$

This shows that $E(\delta) \in B(Z)$, with operator norm ≤ 1.

By (11) and (7), since $E(\delta)x \in Z$ for $x \in Z$,

$$\int_{\mathbb{R}} \frac{1}{\lambda - u} \chi_\delta(u) E(du)x = R(\lambda; A)E(\delta)x = \int_{\mathbb{R}} \frac{1}{\lambda - u} E(du)E(\delta)x.$$

The uniqueness property of the Stieltjes transform implies that

$$E(\sigma)E(\delta)x = \int_{\mathbb{R}} \chi_\sigma(u)\chi_\delta(u)E(du)x = E(\sigma \cap \delta)x$$

for all $\sigma, \delta \in \mathcal{B}(\mathbb{R})$ and $x \in Z$. We have thus shown that E is a spectral measure on Z.

We prove now Statement 2 in the theorem. The argument yielding (4) in the proof of Theorem 1.49 shows that

$$D(A_Z) = R(\lambda; A)Z \tag{13}$$

for any $\lambda \in \mathbb{C} - \mathbb{R}$.

Let $x \in D(A_Z)$. Write then $x = R(\lambda; A)y$ for a fixed nonreal λ and a suitable $y \in Z$. For $-\infty < a < b < \infty$, we have by (11)

$$\begin{aligned} \int_a^b uE(du)x &= \int_a^b uE(du)R(\lambda; A)y \\ &= \int_a^b \frac{u}{\lambda - u} E(du)y \to \int_{\mathbb{R}} \frac{u}{\lambda - u} E(du)y \end{aligned}$$

as $a \to -\infty$ and $b \to \infty$. Thus $\int_{\mathbb{R}} uE(du)x$ exists. Writing $\frac{u}{\lambda-u} = \frac{\lambda}{\lambda-u} - 1$, the last relation shows that

$$\int_{\mathbb{R}} uE(du)x = \lambda \int_{\mathbb{R}} \frac{1}{\lambda - u} E(du)y - E(\mathbb{R})y$$
$$= \lambda R(\lambda; A)y - y = AR(\lambda; A)y = Ax \in Z \tag{14}$$

(since $x \in D(A_Z)$). Thus $D(A_Z) \subset Z_1$, where Z_1 denotes the set on the right-hand side of Statement 2. On the other hand, if $x \in Z_1$, denote $z = \int_{\mathbb{R}} uE(du)x$; we have $z \in Z$, and for nonreal λ, we obtain from (11)

$$R(\lambda; A)z = \lim_{(a,b)} \int_a^b uR(\lambda; A)E(du)x$$
$$= \lim_{(a,b)} \int_a^b \frac{u}{\lambda - u} E(du)x = \int_{\mathbb{R}} \frac{u}{\lambda - u} E(du)x$$
$$= \lambda \int_{\mathbb{R}} \frac{1}{\lambda - u} E(du)x - x = \lambda R(\lambda; A)x - x.$$

Hence

$$x = R(\lambda; A)[\lambda x - z] \in R(\lambda; A)Z = D(A_Z),$$

so that $D(A_Z) = Z_1$. By (14), $Ax = \int_{\mathbb{R}} uE(du)x$ for all $x \in D(A_Z)$.

Finally, let W and F be as in the statement of the theorem. For $x \in W$, Statement 3 (with F replacing E) implies Condition 1 in Definition 2.1. Also, for all $x^* \in X^*$,

$$\|x^* P(\cdot, s)x\|_1 = \frac{s}{\pi} \int_{\mathbb{R}} \left| x^* \int_{\mathbb{R}} \frac{1}{(t-u)^2 + s^2} F(du)x \right| dt$$
$$\leq \int \frac{s}{\pi} \int \frac{dt}{(t-u)^2 + s^2} |x^* Fx|(du) = |x^* Fx|(\mathbb{R}),$$

so that Condition 2 in Definition 2.1 is satisfied as well, i.e., $x \in Z$. Thus $W \subset Z$, and the uniqueness property of the Stieltjes transform implies that $F(\cdot)x = E(\cdot)x$ for $x \in W$. □

The discussion preceding Corollary 1.50 yields the following

Corollary 2.4. *Let A be an operator with real spectrum, acting in the reflexive Banach space X, and let Z be its semi-simplicity space. Then $Z = X$ if and only if A is a scalar-type spectral operator. When this is the case, E is the resolution of the identity for A.*

We consider now the operational calculus τ induced by the spectral measure on Z, E, as defined in the paragraphs following Definition 1.48. (In the following, the space Z is normed by $\|\cdot\|_A$.)

Theorem 2.5. *The map τ is a norm-decreasing algebra homomorphism of $\mathbb{B}(\mathbb{R})$ into $B(Z)$. Moreover, for each $h \in \mathbb{B}(\mathbb{R})$, $\tau(h)$ maps $D(A_Z)$ into itself, and*

$$A\tau(h)x = \tau(h)Ax = \int_{\mathbb{R}} uh(u)E(du)x \quad (x \in D(A_Z)). \tag{15}$$

(Note that the "improper" integral appearing in (15) is $\tau(uh(u))$, defined as usual for functions in $\mathbb{B}_{loc}(\mathbb{R})$.)

Proof. For $x \in Z$ and $h \in \mathbb{B}(\mathbb{R})$, we have by (11)

$$R(\lambda; A)\tau(h)x = \int_{\mathbb{R}} h(u)R(\lambda; A)E(du)x = \int_{\mathbb{R}} \frac{h(u)}{\lambda - u}E(du)x. \tag{16}$$

Therefore

$$\|R(\lambda; A)\tau(h)x\| \leq \frac{\|h\|_\infty \|x\|_A}{|\Im\lambda|} \quad (\lambda \in \mathbb{C} - \mathbb{R}).$$

In particular, $\tau(h)x$ satisfies Condition 1 in Definition 2.1.

By (16), for all $x^* \in X^*$,

$$\begin{aligned}
\|x^*P(\cdot, s)\tau(h)x\|_1 &= \|x^* \int_{\mathbb{R}} p(\cdot - u, s)h(u)E(du)x\|_1 \\
&\leq \int\int |h(u)|p(t-u, s)|x^*Ex|(du)\,dt \\
&\leq \|h\|_\infty \int\int p(t-u, s)\,dt|x^*Ex|(du) \\
&= \|h\|_\infty |x^*Ex|(\mathbb{R}) \leq \|h\|_\infty \|x\|_A \|x^*\|,
\end{aligned}$$

where $|x^*Ex|$ denotes the total variation measure of $x^*Ex = \mu(\cdot; x, x^*)$.

Thus $\tau(h)x$ satisfies Condition 2 in Definition 2.1, and we conclude that $\tau(h)x \in Z$ for all $x \in Z$ and $h \in \mathbb{B}(\mathbb{R})$, and moreover

$$\|\tau(h)x\|_A \leq \|h\|_\infty \|x\|_A \quad (h \in \mathbb{B}(\mathbb{R}), x \in Z). \tag{17}$$

This establishes that τ is norm-decreasing from $\mathbb{B}(\mathbb{R})$ into $B(Z)$. Since E is a spectral measure on Z, it follows that τ is multiplicative on the simple Borel functions, hence on $\mathbb{B}(\mathbb{R})$ as well.

Let $x \in D(A_Z)$. Then $x = R(\lambda; A)y$ for some nonreal λ and $y \in Z$. Therefore, for any $h \in \mathbb{B}(\mathbb{R})$,

$$\tau(h)x = R(\lambda; A)\tau(h)y \in R(\lambda; A)Z = D(A_Z),$$

i.e., $\tau(h)D(A_Z) \subset D(A_Z)$.

In particular, for h, x, y as before, by multiplicativity of τ on $\mathbb{B}(\mathbb{R})$, the limit

$$\lim_{(a,b)} \int_a^b uE(du)\tau(h)x = \lim_{(a,b)} \int_a^b uh(u)E(du)x$$

exists in X and belongs to Z, and equals $A\tau(h)x$ (by Theorem 2.3). Finally, we observe that the bounded operator $AR(\lambda; A) = \lambda R(\lambda; A) - I$ commutes with E, hence with $\tau(h)$, and therefore

$$A\tau(h)x = AR(\lambda; A)\tau(h)y = \tau(h)AR(\lambda; A)y = \tau(h)Ax. \qquad \square$$

Taking in particular the functions $h_t(u) = e^{itu}$ $(t, u \in \mathbb{R})$, let

$$T(t) = \tau(h_t) \quad (t \in \mathbb{R}).$$

Then $T(\cdot)$ is a group of contractions in Z, which map $D(A_Z)$ into itself. It is continuous with respect to the X-norm (as follows at once by dominated convergence), but not necessarily with respect to the Z-norm.

Consider the continuous functions on $\mathbb{R}$

$$k_t(u) = \frac{e^{itu} - 1}{itu} \quad (t, u \neq 0),$$

and $k_t(0) = 1$ (for $t \neq 0$). We have $|k_t(u)| \leq 1$ and $k_t(u) \to 1$ as $t \to 0$ (for all $u \in \mathbb{R}$). For $x \in D(A_Z)$, we apply (15) and the Dominated Convergence Theorem for vector measures:

$$t^{-1}[T(t)x - x] = i \int_{\mathbb{R}} uk_t(u)E(du)x = i\tau(k_t)Ax$$
$$= i \int_{\mathbb{R}} k_t(u)E(du)Ax \to iAx$$

as $t \to 0$ (limit in X). Also

$$s^{-1}[T(t+s)x - T(t)x] = s^{-1}[T(s)T(t)x - T(t)x] \to iAT(t)x$$

as $s \to 0$ (limit in X), since $T(t)x \in D(A_Z)$ for $x \in D(A_Z)$, by Theorem 2.5. We formalize the above discussion in

Corollary 2.6. *$T(\cdot)$ is a group of contractions in Z, continuous in the X-topology on Z, and leaving $D(A_Z)$ invariant. Moreover, in that topology, the generator of $T(\cdot)$ coincides with iA on $D(A_Z)$, and $u := T(\cdot)x$ solves the Abstract Cauchy Problem on $\mathbb{R}$:*

$$u' = iAu, \quad u(0) = x$$

for $x \in D(A_Z)$.

The basic properties of the operational calculus τ on $\mathbb{B}_{loc}(\mathbb{R})$ are collected in the following

Theorem 2.7.

(i) $\tau(\lambda h) = \lambda\tau(h) \quad (0 \neq \lambda \in \mathbb{C}, h \in \mathbb{B}_{loc}(\mathbb{R}))$;

(ii) $D(\tau(h) + \tau(g)) = D(\tau(h+g)) \cap D(\tau(g))$, *and*

$$\tau(h+g)x = \tau(h)x + \tau(g)x$$

for all $x \in D(\tau(h) + \tau(g))$ *and* $h, g \in \mathbb{B}_{loc}(\mathbb{R})$;

(iii) $E(\delta)D(\tau(h)) \subset D(\tau(h))$, *and*

$$\tau(h)E(\delta)x = E(\delta)\tau(h)x = \tau(h\chi_\delta)x \quad (x \in D(\tau(h))),$$

for all compact $\delta \subset \mathbb{R}$ *and* $h \in \mathbb{B}_{loc}(\mathbb{R})$;

(iv) $D(\tau(h)\tau(g)) = D(\tau(hg)) \cap D(\tau(g))$, *and*

$$\tau(h)\tau(g)x = \tau(hg)x$$

for all $x \in D(\tau(h)\tau(g))$ *and* $h, g \in \mathbb{B}_{loc}(\mathbb{R})$.

Proof. (i) is trivial.

In the following, h, g will denote arbitrary functions in $\mathbb{B}_{loc}(\mathbb{R})$.

Proof of (ii). Let $x \in D(\tau(h) + \tau(g)) := D(\tau(h)) \cap D(\tau(g))$. Then

$$\lim_{a,b} \int_a^b h(u)E(du)x$$

exists in X and belongs to Z, and similarly for g. Therefore $\lim_{a,b}$ of the sum of the two integrals (i.e., of $\int_a^b (h+g)(u)E(du)x$) exists and belongs to Z. Thus $x \in D(\tau(h+g))$ and $\tau(h+g)x = \tau(h)x + \tau(g)x$. On the other hand, if $x \in D(\tau(h+g)) \cap D(\tau(g))$, then writing $\int_a^b h(u)E(du)x = \int_a^b (h+g)(u)E(du)x - \int_a^b g(u)E(du)x$, we see that $x \in D(\tau(h))$, so we have the wanted equality of domains.

Proof of (iii). Let $\delta \subset \mathbb{R}$ be compact, $h \in \mathbb{B}_{loc}(\mathbb{R})$, and $x \in D(\tau(h))$. In particular, $x \in Z$, and therefore $E(\delta)x \in Z$. Since τ is multiplicative on $\mathbb{B}(\mathbb{R})$, we have

$$\int_a^b h(u)E(du)E(\delta)x = \tau(h\chi_{[a,b]})\tau(\chi_\delta)x = \tau(h\chi_{[a,b]\cap\delta})$$

$$= \int_{\mathbb{R}} h(u)\chi_{[a,b]\cap\delta}(u)E(du)x.$$

In the last integral, the integrand is majorized by the bounded function $|h|\chi_\delta$, and converges pointwise to $h\chi_\delta$ when $a \to -\infty$ and $b \to \infty$. By dominated convergence for vector measures, it follows that $\lim_{a,b}$ of that integral exists in X and equals $\tau(h\chi_\delta)x \in Z$ (since $h\chi_\delta \in \mathbb{B}(\mathbb{R})$). Hence $E(\delta)x \in D(\tau(h))$ and $\tau(h)E(\delta)x = \tau(h\chi_\delta)x$.

We also have $\tau(h)x \in Z$, because $x \in D(\tau(h))$. By (11), it follows that for $\lambda \in \mathbb{C} - \mathbb{R}$

$$\int_{\mathbb{R}} \frac{1}{\lambda - u} E(du)\tau(h)x = R(\lambda; A)\tau(h)x = R(\lambda; A) \lim_{a,b} \int_a^b h(u)E(du)x$$

$$= \lim_{a,b} \int_a^b h(u)E(du)R(\lambda; A)x$$

$$= \lim_{a,b} \int_a^b \frac{h(u)}{\lambda - u} E(du)x = \int_{\mathbb{R}} \frac{h(u)}{\lambda - u} E(du)x.$$

Hence

$$E(\delta)\tau(h)x = \int_{\mathbb{R}} h(u)\chi_\delta(u)E(du)x = \tau(h\chi_\delta)x,$$

by the uniqueness property of the Stieltjes transform. This completes the proof of (iii).

Proof of (iv). Let $x \in D(\tau(h)\tau(g))$, i.e., $x \in D(\tau(g))$ and $\tau(g)x \in D(\tau(h))$. By the multiplicativity of τ on $\mathbb{B}(\mathbb{R})$ and by (iii),

$$\int_a^b h(u)g(u)E(du)x = \tau(h\chi_{[a,b]}\, g\chi_{[a,b]})x$$

$$= \tau(h\chi_{[a,b]})\tau(g\chi_{[a,b]})x = \tau(h\chi_{[a,b]})E([a,b])\tau(g)x$$

$$= \tau(h\chi_{[a,b]})\tau(g)x \to \tau(h)\tau(g)x$$

when $a \to -\infty$ and $b \to \infty$, since $\tau(g)x \in D(\tau(h))$. Hence $x \in D(\tau(hg))$ and $\tau(hg)x = \tau(h)\tau(g)x$.

In particular, $D(\tau(h)\tau(g)) \subset D(\tau(hg)) \cap D(\tau(g))$.

On the other hand, if x belongs to the right-hand side of the last relation, we have in particular $\tau(g)x \in Z$. Therefore, by the multiplicativity of τ on $\mathbb{B}(\mathbb{R})$ and by (iii), we have

$$\tau(h\chi_{[a,b]})\tau(g)x = \tau(h\chi_{[a,b]}^2)\tau(g)x$$

$$= \tau(h\chi_{[a,b]})E([a,b])\tau(g)x = \tau(h\chi_{[a,b]})\tau(g\chi_{[a,b]})x$$

$$= \tau(hg\chi_{[a,b]})x \to \tau(hg)x \in Z$$

as $a \to -\infty$ and $b \to \infty$, because $x \in D(\tau(hg))$. Hence $\tau(g)x \in D(\tau(h))$. □

We show next that τ operates in the desired way on polynomials.

If $p(u) = \sum_0^n \alpha_k u^k$ with $n \geq 1$ and $\alpha_n \neq 0$, we define as usual

$$p(A_Z) := \sum_0^n \alpha_k A_Z^k = \sum_0^n \alpha_k A^k$$

restricted to

$$D(p(A_Z)) = D(A_Z^n) := \{x \in D(A_Z^{n-1}); A_Z^{n-1}x \in D(A_Z)\}.$$

Theorem 2.8.

1. $D(p(A_Z)) = \bigcap_{k=1}^n D(\tau(u^k))$;
2. $p(A_Z)x = \tau(p)x$ *for all* $x \in D(p(A_Z))$.

Proof. We first prove the following

Lemma. *For* $n = 1, 2, \ldots$ *and any* $\lambda \in \rho(A)$,

(i) $D(A_Z^n) = \{x \in D(A^n); A^k x \in Z, k = 0, 1, \ldots, n\}$;
(ii) $D(A_Z^n) = R(\lambda; A)^n Z$.

Proof of Lemma. (i) is easily verified by induction. The validity of (ii) for $n = 1$ was observed before. Assume (ii) for $n - 1$ (where $n \geq 2$). Since Z is $R(\lambda; A)$-invariant,

$$R(\lambda; A)^n Z \subset R(\lambda; A)^{n-1} Z = D(A_Z^{n-1}),$$

by the induction hypothesis.

Let $x \in R(\lambda; A)^n Z$; then $x \in D(A_Z^{n-1})$, and writing $x = R(\lambda; A)^n y$ with $y \in Z$, we have

$$A^{n-1}x = [AR(\lambda; A)]^{n-1} R(\lambda; A)y = [\lambda R(\lambda; A) - I]^{n-1} R(\lambda; A)y$$

$$= R(\lambda; A)[\lambda R(\lambda; A) - I]^{n-1} y \in D(A) \cap Z.$$

Hence $x \in D(A^n)$ and

$$A^n x = [AR(\lambda; A)]^n y = [\lambda R(\lambda; A) - I]^n y \in Z.$$

By (i), this shows that $x \in D(A_Z^n)$.

On the other hand, if $x \in D(A_Z^n)$, then by (i), $x \in D(A^n)$ and $A^k x \in Z$ for $k = 0, \ldots, n$. Therefore $y := (\lambda I - A)^n x \in Z$, and $x = R(\lambda; A)^n y \in R(\lambda; A)^n Z$ (for $\lambda \in \rho(A)$), and (ii) follows for n. □

Back to the proof of the theorem, let $x \in D(A_Z^n)$ and fix $\lambda \in \rho(A)$. By the lemma, write $x = R(\lambda; A)^n y$ with $y \in Z$. Applying (11) (in the proof of Theorem 2.3) repeatedly, we obtain

$$E(du)x = E(du)R(\lambda; A)^k [R(\lambda; A)^{n-k} y] = (\lambda - u)^{-k} E(du)[R(\lambda; A)^{n-k} y]$$

for $k = 1, \ldots, n$, since $R(\lambda; A)^{n-k} y \in Z$.

Hence, for $-\infty < a < b < \infty$,

$$\int_a^b u^k E(du)x = \int_a^b \left(\frac{u}{\lambda - u}\right)^k E(du)[R(\lambda; A)^{n-k} y]$$

$$\to \int_{\mathbb{R}} \left(\frac{u}{\lambda - u}\right)^k E(du)[R(\lambda; A)^{n-k} y] \tag{18}$$

as $a \to -\infty$ and $b \to \infty$, since $[u/(\lambda - u)]^k$ is a bounded function of u on $\mathbb{R}$.

Thus $\int_{\mathbb{R}} u^k E(du)x$ exists (in X) and equals the integral in (18), which belongs to Z for $k = 1, \dots, n$, by Theorem 2.5. This proves the inclusion $\subset$ in Statement 1 of the theorem.

Next, let x belong to the set on the right-hand side of Statement 1. For each $k = 0, \dots, n$, denote

$$z_k = \int_{\mathbb{R}} u^k E(du)x \quad (\in Z).$$

By (11), we have for $k = 1, \dots, n$,

$$\begin{aligned} R(\lambda; A)z_k &= \lim_{a,b} \int_a^b u^k R(\lambda; A)E(du)x = \lim_{a,b} \int_a^b u^{k-1} \frac{u}{\lambda - u} E(du)x \\ &= \lim_{a,b} \int_a^b u^{k-1} \left(\frac{\lambda}{\lambda - u} - 1\right) E(du)x \\ &= \lim_{a,b} [\lambda R(\lambda; A) - I] \int_a^b u^{k-1} E(du)x = [\lambda R(\lambda; A) - I] z_{k-1}. \end{aligned}$$

Therefore

$$z_{k-1} = R(\lambda; A)(\lambda z_{k-1} - z_k) \in D(A) \cap Z,$$

and

$$A z_{k-1} = [\lambda R(\lambda; A) - I](\lambda z_{k-1} - z_k) = \lambda R(\lambda; A) z_k - [\lambda R(\lambda; A) - I] z_k = z_k,$$

for $k = 1, \dots, n$.

Since $z_0 = x$, it follows from the above recursion that $x \in D(A^n)$ and $A^k x = z_k \in Z$ for $k \le n$, i.e., $x \in D(A_Z^n)$ by Part (i) of the lemma. This proves Statement 1 of the theorem, and also the relation

$$A^k x = \int_{\mathbb{R}} u^k E(du)x \quad (x \in D(A_Z^k), \quad k = 1, 2, \dots), \tag{19}$$

which clearly implies Statement 2. □

A.2 The Case $\mathbb{R}^+ \subset \rho(-A)$

We generalize the construction of the semi-simplicity space to operators $-A$ with spectrum in a halfplane, say, in the closed left halfplane, to fix the ideas. Actually, all we need for our construction is that $\mathbb{R}^+ := (0, \infty)$ be contained in the resolvent set of $-A$. While the Poisson integral representation was the key to the preceding construction, the present one will be based on a theorem of Widder on the Stieltjes integral representation of functions.

Let then

$$R(t) := R(t; -A) \quad (t > 0),$$

and

$$S := AR(I - AR).$$

The function $S(t) = tR(t)[I - tR(t)]$ is a well-defined $B(X)$-valued function on $\mathbb{R}^+$, and for all $k = 1, 2, \ldots$, the powers S^k are of class C^∞.

In the following discussion, the $L^1(\mathbb{R}^+, \frac{dt}{t})$-norm is denoted by $\|\cdot\|_1$.

The Beta function is

$$B(s, t) := \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \quad (s, t \in \mathbb{R}^+).$$

Definition 2.9. *Let $-A$ be an operator with $(0, \infty) \subset \rho(-A)$, and let S be the operator function defined above. The semi-simplicity space for $-A$ is the set Z of all $x \in X$ such that*

$$\sup_{k \in \mathbb{N}} \frac{\|x^* S^k x\|_1}{B(k, k)} < \infty$$

for all $x^* \in X^*$.

Using the Closed Graph Theorem, Fatou's lemma, and the Uniform Boundedness Theorem as in the proof of Lemma 2.2, we obtain

Lemma 2.10. *For all $x \in Z$,*

$$\|x\|_Z := \sup \left\{ \frac{\|x^* S^k x\|_1}{B(k, k)}, \|x\|; k \in \mathbb{N}, \|x^*\| = 1 \right\} < \infty.$$

Lemma 2.11. *The space $Z := (Z, \|\cdot\|_Z)$ is a Banach subspace of X, invariant for any $U \in B(X)$ commuting with A, and $\|U\|_{B(Z)} \leq \|U\|_{B(X)}$.*

Proof. The proof is analogous to the one we gave for the real-spectrum case (see proof of Theorem 2.3). □

(In the following, Z stands for the Banach subspace $(Z, \|\cdot\|_Z)$.)

Theorem 2.12. *Let $-A$ be an operator in the reflexive Banach space X, whose resolvent set contains the axis $\mathbb{R}^+$, and let Z be its semi-simplicity space. Then there exists a spectral measure on Z,*

$$E : \mathcal{B}(\mathbb{R}^+) \to B(Z),$$

such that

1. *for each $\delta \in \mathcal{B}(\mathbb{R}^+)$, $E(\delta)$ commutes with every $U \in B(X)$ which commutes with A;*
2. (i) $D(A_Z) = \{x \in Z; \lim_{b\to\infty} \int_0^b sE(ds)x$ *exists in X and belongs to $Z\}$, and*
   (ii) $Ax = \int_0^\infty sE(ds)x \quad (x \in D(A_Z))$,
   where the last integral is defined as the limit in (i);
3. $R(t)x = \int_0^\infty \frac{1}{t+s} E(ds)x \quad (x \in Z, t > 0)$. *Moreover, Z is "maximal-unique" relative to Property 3, in the sense of Theorem 2.3.*

Proof. Let L_k be the Widder formal differential operators

$$L_k := c_k M^{k-1} D^{2k-1} M^k \quad (k \in \mathbb{N}),$$

where

$$M : f(t) \to tf(t); \quad D : f \to f'$$

are respectively the "multiplication" and the differentiation operators acting on functions of $t \in \mathbb{R}^+$. The constants c_k are given by $c_1 = 1$ and

$$c_k = \frac{(-1)^{k-1}}{\Gamma(k-1)\Gamma(k+1)} \quad (k \geq 2).$$

By Leibnitz' rule,

$$L_k = c'_k \sum_{j=0}^{k} \Gamma(k+j)^{-1} \binom{k}{j} M^{k+j-1} D^{k+j-1},$$

where $c'_1 = 1$ and $c'_k = (-1)^{k-1} B(k-1, k+1)^{-1}$ for $k \geq 2$. Since

$$D^{k+j-1}(x^* Rx) = (-1)^{k+j-1} \Gamma(k+j) x^* R^{k+j} x,$$

we have

$$L_k x^* R(t)x = c''_k t^{-1} x^* (tR)^k \sum_{j=0}^{k} \binom{k}{j} (-tR)^j x = c''_k t^{-1} x^* S^k(t)x,$$

where $c''_1 = 1$ and $c''_k = B(k-1, k+1)^{-1}$ for $k \geq 2$. Therefore, for $x \in Z$ and $x^* \in X^*$,

$$\int_0^\infty |L_k(x^* Rx)|\, dt = c''_k \|x^* S^k x\|_1 \leq \|x\|_Z \|x^*\|, \tag{1}$$

trivially for $k=1$, and because

$$\frac{B(k,k)}{B(k-1,k+1)} = \frac{k-1}{k} < 1$$

for $k>1$.

We now rely on the following complex version of *Widder's theorem* (cf. [W], Theorem 16, p. 361): □

Let f be a C^∞ complex function on $\mathbb{R}^+$, such that

$$K := \sup_{k\in\mathbb{N}} \int_0^\infty |L_k f|\, dt < \infty.$$

Then the limit $c = \lim_{t\to 0+} tf(t)$ exists, and there exists a unique complex regular Borel measure μ on $\mathbb{R}^+$ such that $\|\mu\| \le 2K + |c|$ and

$$f(t) = \int_0^\infty \frac{\mu(ds)}{t+s} \quad (t \in \mathbb{R}^+).$$

Taking $f = x^*Rx$ with $x \in Z$ and $x^* \in X^*$ fixed, we have $K \le \|x\|_Z \|x^*\| < \infty$ by (1). Thus $\lim_{t\to 0+} x^* tR(t)x$ exists for each $x^* \in X^*$. By (a consequence of) the Uniform Boundedness Theorem,

$$\sup_{0<t\le 1} \|tR(t)x\| < \infty$$

for all $x \in Z$. Consider the family of operators $\{tR(t); 0 < t \le 1\} \subset B(Z,X)$ (cf. Lemma 2.11). By the Uniform Boundedness Theorem, it follows from the last relation that

$$H_0 := \sup_{0<t\le 1} \|tR(t)\|_{B(Z,X)} < \infty.$$

Then for each $x \in Z$ and $x^* \in X^*$, the constant c in Widder's theorem for "our" f satisfies the inequality

$$|c| = |c(x,x^*)| := |\lim_{t\to 0+} x^* tR(t)x| \le H_0\, \|x\|_Z \|x^*\|. \tag{2}$$

Let $H := H_0 + 2$. By Widder's theorem and relations (1) and (2), there exists a unique complex regular Borel measure $\mu(\cdot; x, x^*)$ such that

$$\|\mu(\cdot; x, x^*)\| \le H\, \|x\|_Z \|x^*\|$$

and

$$x^* R(t) x = \int_0^\infty \frac{\mu(ds; x, x^*)}{t+s} \quad (t \in \mathbb{R}^+),$$

for all $x \in Z$ and $x^* \in X^*$.

This implies in particular that

$$\|tR(t)x\| \le H\,\|x\|_Z \quad (t \in \mathbb{R}^+). \tag{2'}$$

The uniqueness of the Stieltjes transform implies that for each fixed $\delta \in \mathcal{B}(\mathbb{R}^+)$ and $x \in Z$, $\mu(\delta; x, \cdot)$ is a continuous linear functional on X^*, so that, by reflexivity of X, there exists a unique function $E(\cdot)x : \mathcal{B}(\mathbb{R}^+) \to X$ (for each fixed $x \in Z$) such that

$$\mu(\cdot; x, x^*) = x^* E(\cdot)x \quad (x^* \in X^*).$$

Necessarily, $E(\delta)$ is a linear operator with domain Z, and

$$\|E(\delta)x\| \le H\,\|x\|_Z \quad (\delta \in \mathcal{B}(\mathbb{R}^+), x \in Z).$$

By Pettis' theorem, $E(\cdot)x$ is a strongly countably additive vector measure (in X), and

$$R(t)x = \int_0^\infty \frac{E(ds)x}{t+s} \quad (t > 0, x \in Z).$$

This is Property 3, which corresponds to (7) in the proof of Theorem 2.3. As in the latter case, we shall see that it implies that E is a spectral measure on Z satisfying Properties 1 and 2 of our theorem.

Property 1 is an immediate consequence of the uniqueness property of the Stieltjes transform. Taking then, in particular, $U = R(u)$ for $u > 0$ fixed, we obtain for $x \in Z$

$$\begin{aligned} R(u)E(\mathbb{R}^+)x = E(\mathbb{R}^+)R(u)x &= \lim_{t\to\infty} \int_0^\infty \frac{t}{t+s} E(ds)R(u)x \\ &= \lim_t tR(t)R(u)x = \lim_t \frac{t}{t-u} R(u)x - \lim_t \frac{tR(t)x}{t-u} = R(u)x, \end{aligned}$$

by the resolvent equation and (2′). Since $R(u)$ is one-to-one, it follows that $E(\mathbb{R}^+) = I|_Z$.

For $t, u > 0, t \ne u$, and $x \in Z$, we have by Property 3, the resolvent equation, and the fact that $R(u)x \in Z$,

$$\begin{aligned} \int_0^\infty \frac{1}{t+s} E(ds)R(u)x &= R(t)R(u)x \\ &= \frac{1}{t-u} \int_0^\infty \left[\frac{1}{u+s} - \frac{1}{t+s}\right] E(ds)x \\ &= \int_0^\infty \frac{1}{t+s} \left[\frac{1}{u+s} E(ds)x\right]. \end{aligned}$$

By the uniqueness of the Stieltjes transform,

$$E(ds)R(u)x = \frac{1}{u+s} E(ds)x, \tag{3}$$

and inductively,

$$E(ds)R(u)^k x = \frac{1}{(u+s)^k} E(ds)x,$$

for all $k \in \mathbb{N}, u > 0$, and $x \in Z$. Therefore

$$E(ds)p(R(u))x = p\left(\frac{1}{u+s}\right) E(ds)x$$

for all polynomials p. In particular,

$$\begin{aligned} E(ds)S^k(u)x &= \frac{u^k}{(u+s)^k}\left[1 - \frac{u}{u+s}\right]^k E(ds)x \\ &= \frac{(us)^k}{(u+s)^{2k}} E(ds)x, \end{aligned}$$

for all $u > 0, k \in \mathbb{N}$, and $x \in Z$. Property 1 for $U = S^k(u)$ implies then that

$$x^* S^k(u)E(\delta)x = \int_\delta \frac{(us)^k}{(u+s)^{2k}} x^* E(ds)x.$$

By Tonelli's theorem, for all $x \in Z$, $x^* \in X^*$, and $\delta \in \mathcal{B}(\mathbb{R}^+)$, we have

$$\begin{aligned} \|x^* S^k E(\delta)x\|_1 &\le \int_\delta \int_0^\infty \frac{(us)^k}{(u+s)^{2k}} \frac{du}{u} |x^* E(\cdot)x|(ds) \\ &= \int_\delta \int_0^\infty \frac{t^k}{(1+t)^{2k}} \frac{dt}{t} |x^* E(\cdot)x|(ds) = B(k,k)|x^* Ex|(\delta) \\ &\le B(k,k)\, H\, \|x\|_Z \|x^*\|, \end{aligned}$$

therefore $E(\delta)x \in Z$ and $\|E(\delta)x\|_Z \le H\,\|x\|_Z$, that is, $E(\delta) \in B(Z)$ and

$$\|E(\delta)\|_{B(Z)} \le H \quad (\delta \in \mathcal{B}(\mathbb{R}^+)). \tag{4}$$

Using Property 3 with the vector $E(\delta)x \in Z$ (whenever $x \in Z$), Property 1 (with $U = R(u)$ for any $u > 0$), and relation (3), it follows that

$$\begin{aligned} R(u)E(\delta)x &= \int_0^\infty \frac{1}{u+s} E(ds)E(\delta)x \\ &= E(\delta)R(u)x = \int_0^\infty \frac{1}{u+s} \chi_\delta(s)E(ds)x, \end{aligned}$$

and therefore, by the uniqueness of the Stieltjes transform,

$$E(\sigma)E(\delta)x = \int_0^\infty \chi_\sigma(s)\chi_\delta(s)E(ds)x = E(\sigma \cap \delta)x$$

for all $\sigma, \delta \in \mathcal{B}(\mathbb{R}^+)$. In conclusion, E is a spectral measure on Z.

Since $D(A_Z) = R(t)Z$ for any $t > 0$, write any given $x \in D(A_Z)$ as $x = R(t)y$ for a fixed $t > 0$ and a suitable $y \in Z$. Then as $b \to \infty$

$$\int_0^b sE(ds)x = \int_0^b \frac{s}{t+s}E(ds)y \to \int_0^\infty \frac{s}{t+s}E(ds)y$$
$$= \int_0^\infty \left[1 - \frac{t}{t+s}\right] E(ds)y = [I - tR(t)]y (\in Z)$$
$$= AR(t)y = Ax.$$

If Z_1 denotes the set on the right-hand side of Property 2(i), we obtained that $D(A_Z) \subset Z_1$ and Property 2(ii) is valid on $D(A_Z)$. On the other hand, if $x \in Z_1$, denote the limit in Property 2(i) by $z \in Z$. Then for any $t > 0$,

$$R(t)z = \lim_{b\to\infty} \int_0^b sR(t)E(ds)x = \lim_b \int_0^b \frac{s}{t+s}E(ds)x$$
$$= \int_0^\infty \frac{s}{t+s}E(ds)x = x - tR(t)x.$$

Therefore $x = R(t)[z + tx] \in R(t)Z = D(A_Z)$, so that $D(A_Z) = Z_1$.

Suppose now that W is a Banach subspace of X and F is a spectral measure on W with Property 3 of E. Fix $x \in W$. Differentiating repeatedly, we obtain

$$R^k(t)x = \int_0^\infty \left(\frac{1}{t+s}\right)^k F(ds)x$$

for all $k = 1, 2, \ldots$ and $t > 0$. Therefore

$$p(R(t))x = \int_0^\infty p\left(\frac{1}{t+s}\right) F(ds)x \tag{5}$$

for all polynomials p and $t > 0$. In particular,

$$S^k(t)x = \{tR(t)[1 - tR(t)]\}^k = \int_0^\infty \left\{\frac{t}{t+s}\left[1 - \frac{t}{t+s}\right]\right\}^k F(ds)x$$
$$= \int_0^\infty \frac{(ts)^k}{(t+s)^{2k}}F(ds)x.$$

Therefore, for all $x^* \in X^*$ and $k \in \mathbb{N}$, we have by Tonelli's theorem

$$\frac{\|x^* S^k x\|_1}{B(k,k)} \le B(k,k)^{-1} \int_0^\infty \int_0^\infty \frac{(ts)^k}{(t+s)^{2k}} |x^* F x|(ds) \frac{dt}{t}$$
$$= B(k,k)^{-1} \int_0^\infty \int_0^\infty \frac{u^k}{(1+u)^{2k}} \frac{du}{u} |x^* F x|(ds) = \|x^* F x\| < \infty.$$

Hence $x \in Z$, i.e., $W \subset Z$ (topologically!). Also, for all $x \in W \subset Z$, we have

$$R(t)x = \int_0^\infty \frac{1}{t+s} F(ds)x = \int_0^\infty \frac{1}{t+s} E(ds)x \quad (t > 0),$$

and therefore $F(\delta)x = E(\delta)x$ for all $\delta \in \mathcal{B}(\mathbb{R}^+)$, by the uniqueness property of the Stieltjes transform. □

As before, the important special case $Z = X$ gives the following result.

Theorem 2.13. *Let $-A$ be an operator with $\mathbb{R}^+ \subset \rho(-A)$, acting in the reflexive Banach space X, and let Z be its semi-simplicity space. Then the following statements are equivalent:*

(a) $Z = X$.
(b) $K := \sup_{\|x\|=1} \|x\|_Z < \infty$.
(c) *A is spectral of scalar type, with spectrum in* $[0, \infty)$.

Proof. Since $\|\cdot\| \le \|\cdot\|_Z$, the equivalence of (a) and (b) follows from the Closed Graph Theorem.

Assume now (a). Then E is a spectral measure in the usual sense, and Property 2 of Theorem 2.12 just states that A is a scalar-type spectral operator with resolution of the identity E, and then necessarily $\sigma(A) \subset [0, \infty)$ (cf. [DS I–III]).

We show finally that (c) implies (a) (even without the reflexivity hypothesis). Let E be the resolution of the identity of the scalar-type operator A with spectrum in $[0, \infty)$. Then E is a spectral measure on X satisfying Property 3 of the theorem (on X). By the maximality property of Z (which is valid without the reflexivity hypothesis), we have necessarily $X = Z$. □

The operational calculus results contained in Theorems 2.5, 2.7, and 2.8, and in Corollary 2.6 (with the obvious modification), are generalized in a routine way to the present situation (i.e., with the assumption $\mathbb{R}^+ \subset \rho(-A)$).

B

The Laplace–Stieltjes Space

Observe that Theorem 2.12 applies in particular to the case where $-A$ generates a C_o-semigroup of contractions, $T(\cdot)$. In that case, by (5) in the preceding subsection (for the spectral measure on Z, $E(\cdot)$), we have for all $x \in Z$ and $t > 0$

$$\left[\frac{n}{t} R\left(\frac{n}{t}\right)\right]^n x = \int_0^\infty \left[\frac{n}{t} \frac{1}{\frac{n}{t} + s}\right]^n E(ds)x$$
$$= \int_0^\infty \frac{E(ds)x}{\left[1 + \frac{ts}{n}\right]^n} \to \int_0^\infty e^{-ts} E(ds)x$$

as $n \to \infty$, by the Lebesgue Dominated Convergence Theorem for vector measures (convergence in $X!$). By Theorem 1.36, it follows that

$$T(t)x = \int_0^\infty e^{-ts} E(ds)x \quad (x \in Z, t \geq 0),$$

that is, $T(\cdot)x$ is the Laplace–Stieltjes transform of the vector measure $E(\cdot)x$ (in $X!$) for all $x \in Z$.

We shall consider in this section the more general question of constructing a maximal Banach subspace W of X on which a given family of closed operators in X is the Laplace–Stieltjes transform of an adequate "vector measure on W." The result will be applied in particular to *semigroups of closed operators*. A variant of the construction will produce the so-called *integrated Laplace space* for a given family of closed operators.

B.1 The Laplace–Stieltjes Space

Denote by $\mathcal{L}$ the Laplace transform,

$$(\mathcal{L}\phi)(t) := \int_0^\infty e^{-ts} \phi(s)\, ds \quad (t \geq 0),$$

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acting on a space of functions to be specified as we proceed. We may choose for example the space

$$C_c^\infty := C_c^\infty(\mathbb{R}^+)$$

of all complex C^∞-functions with compact support in $\mathbb{R}^+$.

Let $K(X)$ denote the set of all closed operators acting on X.

Definition 2.14. *Let $F : [0,\infty) \to K(X)$ be such that $F(0) = I$. The Laplace–Stieltjes space for F is the set W of all x in the "common domain" of F,*

$$\mathcal{D} := \bigcap_{s>0} D(F(s)),$$

such that $F(\cdot)x$ is strongly continuous on $[0,\infty)$, and

$$\|x\|_W := \sup\left\{\left\|\int_0^\infty \phi(s)F(s)x\,ds\right\| ; \phi \in C_c^\infty, \|\mathcal{L}\phi\|_\infty = 1\right\}$$

is finite.

Theorem 2.15. *Let W be the Laplace–Stieltjes space for F, normed by $\|\cdot\|_W$. Then W is a Banach subspace of X, and in case X is reflexive, there exists a uniquely determined function E on $\mathcal{B}([0,\infty))$ into the closed unit ball $B(W,X)_1$ of $B(W,X)$, such that*

(i) *for each $x \in W$, $E(\cdot)x$ is a regular countably additive X-valued measure, and*
(ii) *$F(t)x = \int_0^\infty e^{-ts}E(ds)x$ for all $t \geq 0$ and $x \in W$.*
(iii) *If $T \in B(X)$ leaves the common domain $\mathcal{D}$ invariant and commutes with $F(s)|_{\mathcal{D}}$ for all $s > 0$, then $T \in B(W)$ (with $\|T\|_{B(W)} \leq \|T\|_{B(X)}$) and $TE(\delta) = E(\delta)T$ on W, for all $\delta \in \mathcal{B}([0,\infty))$.*

Moreover,the pair (W,E) is maximal-unique in the following sense: if (Y,E') is a pair with the properties (i) and (ii) of (W,E), then $(Y,E') \subset (W,E)$, meaning that Y is continuously embedded in W and $E'(\delta) = E(\delta)|_Y$ for all $\delta \in \mathcal{B}([0,\infty))$.

The proof depends on a general "duality lemma" (that has many other applications as well). It gives a simple criterion for belonging to the range of the adjoint T^* of an arbitrary densely defined operator T.

Lemma 2.16. *Let $\mathcal{E}, \mathcal{F}$ be normed spaces, and let $T : \mathcal{E} \to \mathcal{F}$ be a densely defined linear operator. Let $u^* \in \mathcal{E}^*$ and $M > 0$ be given. Then there exists $v^* \in D(T^*)$ with $\|v^*\| \leq M$ such that $u^* = T^*v^*$ if and only if*

$$|u^*u| \leq M\,\|Tu\| \quad (u \in D(T)). \tag{1}$$

Proof. If $u^* = T^* v^*$ with $v^* \in D(T^*)$ such that $\|v^*\| \le M$, then for all $u \in D(T)$,

$$|u^* u| = |(T^* v^*)(u)| = |v^*(Tu)|$$
$$\le \|v^*\| \, \|Tu\| \le M \, \|Tu\|.$$

Conversely, if (1) is satisfied, define

$$\pi : \mathrm{ran}(T) \to \mathbb{C}$$

by

$$\pi(Tu) = u^* u \quad (u \in D(T)).$$

If $u, u' \in D(T)$ are such that $Tu = Tu'$, then by (1),

$$|u^* u - u^* u'| = |u^*(u - u')| \le M \|T(u - u')\| = 0,$$

so that π is well-defined. It is linear and bounded on $\mathrm{ran}(T)$, with norm $\le M$ (by (1)). By the Hahn–Banach theorem, there exists $v^* \in \mathcal{F}^*$ such that $\|v^*\| \le M$ and

$$v^*|_{\mathrm{ran}(T)} = \pi.$$

Thus

$$v^*(Tu) = u^* u \quad (u \in D(T)).$$

This shows that $v^* \in D(T^*)$ and $T^* v^* = u^*$. □

Note that for $T \in B(\mathcal{E}, \mathcal{F})$, Condition (1) needs to be required only for all u in a *dense* subset of $\mathcal{E}$.

We apply the lemma to the Laplace–Stieltjes transform:

Lemma 2.17. *A function $h : [0, \infty)$ is the Laplace–Stieltjes transform $h(t) = \int_0^\infty e^{-ts} \mu(ds)$ of a regular complex Borel measure μ on $[0, \infty)$ with total variation norm $\|\mu\| \le M$ if and only if it is continuous and*

$$\left| \int_0^\infty h(t) \phi(t) \, dt \right| \le M \, \|\mathcal{L}\phi\|_\infty$$

for all $\phi \in C_c^\infty(\mathbb{R}^+)$).

Proof. If h is the Laplace–Stieltjes transform of some complex Borel measure μ with $\|\mu\| \le M$, it is certainly continuous, so that the integrals $\int_0^\infty h(t)\phi(t) \, dt$ make sense for all $\phi \in C_c^\infty$, and by Fubini's theorem

$$\left| \int_0^\infty h(t) \phi(t) \, dt \right| = \left| \int_0^\infty (\mathcal{L}\phi)(s) \mu(ds) \right|$$
$$\le \|\mu\| \, \|\mathcal{L}\phi\|_\infty \le M \, \|\mathcal{L}\phi\|_\infty.$$

For the converse, apply Lemma 2.16 to the operator

$$\mathcal{L} : L^1([0,\infty)) \to C_0([0,\infty)),$$

where $C_0([0,\infty))$ denotes the space of all complex continuous functions on $[0,\infty)$ vanishing at ∞. Its adjoint space is the space $M([0,\infty))$ of all regular complex Borel measures on $[0,\infty)$ with the total variation norm, and

$$\mathcal{L}^* : M([0,\infty)) \to L^\infty([0,\infty))$$

is the Laplace–Stieltjes transform (by Fubini's theorem). If the given continuous function h satisfies our lemma's condition, then since

$$\|\mathcal{L}\phi\|_\infty \le \|\phi\|_1 := \|\phi\|_{L^1([0,\infty))},$$

we have necessarily $\|h\|_\infty \le M$, i.e., $h \in (L^1)^*$, and by Lemma 2.16, there exists $\mu \in M([0,\infty))$ with $\|\mu\| \le M$ such that $h = \mathcal{L}^*\mu$ (everywhere, by continuity of both sides). □

Lemma 2.18. *The Laplace–Stieltjes space W for F is a Banach subspace of X, and if $T \in B(X)$ leaves $\mathcal{D}$ invariant and commutes with each $F(s)|_{\mathcal{D}}$, then $T \in B(W)$ (with $B(W)$-norm $\le \|T\|$).*

Proof. Clearly, W is a linear manifold in X, and $\|\cdot\|_W$ is a semi-norm on W. If $x \in W$, then

$$\left\| \int_0^\infty \phi(t)F(t)x\,dt \right\| \le \|x\|_W \|\mathcal{L}\phi\|_\infty (\le \|x\|_W \|\phi\|_1) \tag{2}$$

for all $\phi \in C_c^\infty$, hence necessarily

$$\sup_{t\ge 0} \|F(t)x\| \le \|x\|_W \quad (x \in W). \tag{3}$$

In particular, since $x = F(0)x$, $\|x\| \le \|x\|_W$, and therefore the space W with the norm $\|\cdot\|_W$ is a normed subspace of X. We prove its completeness. Let $\{x_n\}$ be Cauchy in W (hence in X), and let x be its X-limit. For $\epsilon > 0$ given, let $n_o \in \mathbb{N}$ be such that $\|x_n - x_m\|_W < \epsilon$ for all $n, m > n_o$. Then for all $\phi \in C_c^\infty$ and $n, m > n_o$,

$$\left\| \int_0^\infty \phi(t)F(t)(x_n - x_m)\,dt \right\| \le \epsilon \|\mathcal{L}\phi\|_\infty \le \epsilon \|\phi\|_1. \tag{4}$$

Hence

$$\|F(t)(x_n - x_m)\| \le \epsilon \quad (n, m > n_o; t \ge 0),$$

that is, $\{F(t)x_n\}$ is uniformly Cauchy in X on $[0,\infty)$. Let then $g(t) := \lim_n F(t)x_n$ (limit in X, uniformly in $t \in [0,\infty)$). Since $x_n \in D(F(t))$ and $x_n \to x$ in X, it follows that $x \in D(F(t))$ and $F(t)x = g(t)$ for each $t \ge 0$, because $F(t)$ is a closed operator. Thus $F(\cdot)x_n \to F(\cdot)x$ uniformly on $[0,\infty)$, so that $F(\cdot)x$ is continuous on $[0,\infty)$ and

$$\int_0^\infty \phi(t)F(t)x_n\,dt \to \int_0^\infty \phi(t)F(t)x\,dt \quad (\phi \in C_c^\infty)$$

strongly in X. Letting $n \to \infty$ in (4), we obtain

$$\left\| \int_0^\infty \phi(t)F(t)(x-x_m)\,dt \right\| \le \epsilon \|\mathcal{L}\phi\|_\infty \quad (m > n_0; \phi \in C_c^\infty).$$

Hence $\|x - x_m\|_W \le \epsilon$ for all $m > n_o$; therefore $x - x_m \in W$ (and so $x = (x-x_m)+x_m \in W$), and $\|x - x_m\|_W \to 0$ when $m \to \infty$. Thus W is complete.

If T is as in the statement of the lemma, then for each $x \in W$, we have $Tx \in \mathcal{D}, F(\cdot)Tx = T[F(\cdot)x]$ is continuous, and for all $\phi \in C_c^\infty$,

$$\begin{aligned}\left\| \int_0^\infty \phi(t)F(t)Tx\,dt \right\| &= \left\| T \int_0^\infty \phi(t)F(t)x\,dt \right\| \\ &\le \|T\|_{B(X)} \|x\|_W \|\mathcal{L}\phi\|_\infty,\end{aligned}$$

so that $TW \subset W$ and $\|T\|_{B(W)} \le \|T\|_{B(X)}$. □

Proof of Theorem 2.15. For each $x \in W$ and $x^* \in X^*$, $x^*F(\cdot)x$ is a complex continuous function on $[0,\infty)$ satisfying

$$\left| \int_0^\infty \phi(t)[x^*F(t)x]\,dt \right| \le \|x\|_W \|\mathcal{L}\phi\|_\infty \|x^*\| \quad (\phi \in C_c^\infty).$$

By Lemma 2.17, there exists a unique $\mu = \mu(\cdot; x, x^*) \in M([0,\infty))$ such that

$$\|\mu(\cdot; x, x^*)\| \le \|x\|_W \|x^*\| \tag{5}$$

and

$$x^*F(t)x = \int_0^\infty e^{-ts}\mu(ds; x, x^*) \tag{6}$$

for all $t \ge 0, x \in W$, and $x^* \in X^*$. The uniqueness of the representation (6) implies the bilinearity of $\mu(\delta; \cdot, \cdot)$ for each fixed $\delta \in \mathcal{B}([0,\infty))$, and since X is reflexive, it follows from (5) that there exists a unique $E(\delta) \in B(W, X)_1$ such that

$$\mu(\delta; x, x^*) = x^*E(\delta)x \tag{7}$$

for all $x \in W, x^* \in X^*$, and $\delta \in \mathcal{B}([0,\infty))$.

Statements (i),(ii),(iii) of the theorem follow now from (7), Pettis' theorem, (6), and Lemma 2.18.

Let (Y, E') be as in the statement of the theorem. Property (ii) for Y contains implicitly the fact that Y is contained in the common domain $\mathcal{D}$ of

$F(\cdot)$. Also if $x \in Y$, $F(\cdot)x$ is X-continuous on $[0,\infty)$ (as the Laplace–Stieltjes transform of the vector measure $E'(\cdot)x$), and by Lemma 2.17,

$$\|x\|_W = \sup\left\{\left|\int_0^\infty \phi(t)x^*F(t)x\,dt\right|; \|x^*\| = 1, \phi \in C_c^\infty, \|\mathcal{L}\phi\|_\infty = 1\right\}$$
$$\leq \sup\{\|x^*E'(\cdot)x\|; \|x^*\| = 1\} := K_x < \infty,$$

that is, $x \in W$. Since $K_x \leq K\|x\|_Y$ for a suitable finite constant K, the inclusion $Y \subset W$ is topological. The fact $E'(\cdot) = E(\cdot)|_Y$ follows from the uniqueness property of the Laplace–Stieltjes transform of regular measures.

□

B.2 Semigroups of Closed Operators

We shall now apply Theorem 2.15 to *semigroups of closed operators.*

Definition 2.19. *The family $\{T(t); t \geq 0\}$ of closed operators is called a semigroup of closed operators if $T(0) = I$, and $T(s)T(t)x = T(s+t)x$ for all x in the common domain $\mathcal{D}$ of $T(\cdot)$.*

Theorem 2.20. *Let $T(\cdot)$ be a semigroup of closed operators on the reflexive Banach space X with common domain $\mathcal{D}$, and let W be its Laplace–Stieltjes space. Then there exists a uniquely determined spectral measure on W,*

$$E : \mathcal{B}([0,\infty)) \to B(W)_1,$$

such that

$$T(t)x = \int_0^\infty e^{-ts}E(ds)x \quad (t \geq 0; x \in W).$$

Moreover, $E(\cdot)$ commutes with every operator $U \in B(X)$ such that $U\mathcal{D} \subset \mathcal{D}$ and $UT(\cdot)x = T(\cdot)Ux$ for all $x \in \mathcal{D}$. Setting

$$\tau(h)x := \int_0^\infty h(s)E(ds)x$$

for $h \in ([0,\infty])$ and $x \in W$, the resulting map

$$\tau : ([0,\infty]) \to B(W)$$

is a norm-decreasing algebra homomorphism.

Proof. Let E be associated with the family $F(\cdot) = T(\cdot)$ as in Theorem 2.15. Let $x \in W$. Then for each $s \geq 0$, $T(\cdot)T(s)x = T(\cdot + s)x$ is strongly continuous on $[0,\infty)$, and for all $\phi \in C_c^\infty := C_c^\infty(\mathbb{R}^+)$, denoting $\phi_s(u) := \phi(u-s)$, we have

$$\left\|\int_0^\infty \phi(t)T(t)[T(s)x]\,dt\right\| = \left\|\int \phi_s(u)T(u)x\,du\right\|$$
$$\leq \|x\|_W\|\mathcal{L}\phi_s\|_\infty \leq \|x\|_W\|\mathcal{L}\phi\|_\infty,$$

since $(\mathcal{L}\phi_s)(t) = e^{-ts}(\mathcal{L}\phi)(t)$, so that $\|\mathcal{L}\phi_s\|_\infty \leq \|\mathcal{L}\phi\|_\infty$. Thus $T(s)x \in W$, and $\|T(s)x\|_W \leq \|x\|_W$. This shows that $T(\cdot)$ is a semigroup of contractions on the Banach subspace W of X (continuous with respect to the X-norm!).

By Theorem 2.15, for $x \in W$ and $t \geq 0$,

$$T(t)\int_0^\infty e^{-su}E(du)x = T(t)T(s)x$$
$$= T(t+s)x = \int_0^\infty e^{-tu}e^{-su}E(du)x.$$

By linearity, this shows that

$$T(t)\tau(h)x = \int_0^\infty e^{-tu}h(u)E(du)x, \tag{1}$$

for all $x \in W$ and $h(u) = \sum_{j=1}^n c_j e^{-s_j u}$ with $c_j \in \mathbb{C}$ and $s_j \geq 0$. These finite linear combinations are dense in $C_b := C_b([0,\infty))$, the space of all bounded continuous complex functions on $[0,\infty)$. If $h \in C_b$, pick a sequence h_k of such combinations such that $h_k \to h$ uniformly on $[0,\infty)$. Then for each $t \geq 0$, $\tau(h_k)x \in D(T(t))$, $\tau(h_k)x \to_k \tau(h)x$, and by (1), $T(t)\tau(h_k)x = \int_0^\infty e^{-tu}h_k(u)E(du)x \to_k \int_0^\infty e^{-tu}h(u)E(du)x$. Since $T(t)$ is closed, it follows that $\tau(h)x \in D(T(t))$ and (1) is valid for all $h \in C_b$. This is easily extended to $h \in \mathbb{B}([0,\infty))$. Indeed, since the vector measure $E(\cdot)x$ is regular (for each $x \in W$), there exists a sequence $\{h_k\} \subset C_b$ such that $\|h_k\|_\infty = \|h\|_\infty$ and $h_k \to h$ pointwise almost everywhere with respect to $E(\cdot)x$. By the Lebesgue Dominated Convergence Theorem for vector measures (cf. [DS I–III], p. 328), $\tau(h_k)x \to \tau(h)x$ in X, $\tau(h_k)x \in D(T(t))$ (for each $t > 0$), and by (1) for h_k, $T(t)\tau(h_k)x \to \int_0^\infty e^{-tu}h(u)E(du)x$. Since $T(t)$ is closed, it follows that $\tau(h)x \in D(T(t))$ and (1) is valid for h.

Thus $\tau(h)x \in \mathcal{D}$, and by (1), we have for all $\phi \in C_c^\infty$,

$$\left\|\int_0^\infty \phi(t)T(t)[\tau(h)x]\,dt\right\| = \left\|\int_0^\infty \phi(t)\int_0^\infty e^{-tu}h(u)E(du)x\,dt\right\|$$
$$= \left\|\int_0^\infty (\mathcal{L}\phi)(u)h(u)E(du)x\right\| \leq \|h\|_\infty\|x\|_W\|\mathcal{L}\phi\|_\infty.$$

(Cf. (5) in the preceding subsection.)

Therefore

$$\|\tau(h)x\|_W \leq \|h\|_\infty\|x\|_W \quad (x \in W, h \in \mathbb{B}([0,\infty)), \tag{2}$$

i.e., τ is a norm-decreasing (linear) map of $\mathbb{B}([0,\infty)$ into $B(W)$.

Taking $h = \chi_\delta$, we have $E(\delta) \in B(W)_1$, and by (1), for $x \in W$, etc.,

$$\int_0^\infty e^{-tu} E(du)[E(\delta)x] = T(t)[E(\delta)x] = \int_0^\infty e^{-tu} \chi_\delta(u) E(du)x.$$

By the uniqueness property of the Laplace–Stieltjes transform of regular measures, it follows that

$$E(du)E(\delta)x = \chi_\delta(u)E(du)x,$$

and therefore

$$E(\sigma)E(\delta)x = E(\sigma \cap \delta)x$$

for all $\sigma, \delta \in \mathcal{B}([0,\infty))$ and $x \in W$. Thus E is a spectral measure on W, and τ is necessarily multiplicative on $\mathbb{B}([0,\infty))$, since it is multiplicative on the simple Borel functions, and satisfies (2) on $\mathbb{B}([0,\infty))$.

In view of Theorem 2.15, this completes the proof of Theorem 2.20. □

In the special case of a C_o-semigroup $T(\cdot)$ of bounded operators, with generator $-A$ such that $\mathbb{R}^+ \subset \rho(-A)$ (for example, in case of a *uniformly bounded* C_o-semigroup), the semi-simplicity space Z for $-A$ is well-defined, as well as the Laplace–Stieltjes space W for $T(\cdot)$. As expected, we have

Theorem 2.21. *Let $-A$ generate the contractions C_o-semigroup $T(\cdot)$ on the reflexive Banach space X. Let Z and W be the semi-simplicity space for $-A$ and the Laplace–Stieltjes space for $T(\cdot)$, respectively. Then $Z = W$, topologically.*

Proof. The observations at the beginning of the section and the maximality of W show that $Z \subset W$.

On the other hand, if $x \in W$, then by Theorem 2.20

$$R(t)x = \int_0^\infty e^{-ts} T(s)x\, ds = \int_0^\infty e^{-ts} \int_0^\infty e^{-su} E(du)x\, ds$$
$$= \int_0^\infty \int_0^\infty e^{-(t+u)s}\, ds E(du)x = \int_0^\infty \frac{1}{t+u} E(du)x,$$

where the change of integration order is easily justified by the Tonelli and Fubini theorems. By the multiplicativity of the map τ induced by E, the spectral measure on W (cf. Theorem 2.20), we get for all $k \in \mathbb{N}$,

$$S^k(t)x := \{tR(t)[1 - tR(t)]\}^k x = \int_0^\infty \left\{ \frac{t}{t+u} \left[1 - \frac{t}{t+u} \right] \right\}^k E(du)x$$
$$= \int_0^\infty \left\{ \frac{tu}{(t+u)^2} \right\}^k E(du)x = \int_0^\infty \left(\frac{u}{t} \right)^k \left(1 + \frac{u}{t} \right)^{-2k} E(du)x.$$

Therefore, for all unit vectors $x^* \in X^*$,

$$\begin{aligned}\|x^* S^k x\|_1 &\leq \int_0^\infty \int_0^\infty \left(\frac{u}{t}\right)^k \left(1 + \frac{u}{t}\right)^{-2k} \frac{dt}{t} |x^* E(\cdot)x|(du) \\ &= \int_0^\infty \int_0^\infty s^k (1+s)^{-2k} \frac{ds}{s} |x^* E(\cdot)x|(du) \\ &= B(k,k) \|x^* E(\cdot)x\| \leq B(k,k)\|x\|_W,\end{aligned}$$

where we used (5) and (7) in the proof of Theorem 2.15.

Hence

$$\|x\|_Z := \sup\left\{\|x\|, \frac{\|x^* S^k x\|_1}{B(k,k)}; k \in \mathbb{N}, \quad \|x^*\| = 1\right\} \leq \|x\|_W < \infty,$$

that is, $W \subset Z$.

We thus proved that $Z = W$, and since both are Banach spaces and $\|\cdot\|_Z \leq \|\cdot\|_W$, it follows that the norms are equivalent (by a well-known theorem of Banach). □

B.3 The Integrated Laplace Space

We consider now a "variation" of the construction of the Laplace–Stieltjes space for a family $F(\cdot)$ as given in Definition 2.14. In that construction, the constraint on $\phi \in C_c^\infty := C_c^\infty(\mathbb{R}^+)$ was $\|\mathcal{L}\phi\|_\infty = 1$. Since ϕ has *compact support* in $\mathbb{R}^+ = (0, \infty)$, $\mathcal{L}\phi \in L^1(\mathbb{R}^+, dt)$ (with the *usual* Lebesgue measure dt), and

$$\|\mathcal{L}\phi\|_{L^1(\mathbb{R}^+,\, dt)} \leq \|\phi\|_{L^1(\mathbb{R}^+,\, ds/s)} < \infty.$$

It then makes sense to replace the norm $\|\cdot\|_\infty$ by the norm

$$\|\cdot\|_1 := \|\cdot\|_{L^1(\mathbb{R}^+,\, dt)}$$

in the constraint on ϕ (in Definition 2.14). The result will be an *integrated Laplace integral representation* for the given family F on an adequate "maximal" Banach subspace.

Definition 2.22. *Let $\{F(t); t \geq 0\}$ be a family of closed operators with common domain $\mathcal{D}$. The integrated Laplace space for $F(\cdot)$ is the set Y of all $x \in \mathcal{D}$ such that $F(\cdot)x$ is X-continuous on $\mathbb{R}^+$, and*

$$\|x\|_Y := \sup\left\{\|x\|, \left\|\int_0^\infty \phi(t) F(t) x \, dt\right\|; \phi \in C_c^\infty, \|\mathcal{L}\phi\|_1 = 1\right\}$$

is finite.

Theorem 2.23. *Let Y be the integrated Laplace space for the family $F(\cdot)$ of closed operators on the (arbitrary) Banach space X. Then Y is a Banach subspace of X, invariant for every $T \in B(X)$ commuting with $F(t)|_{\mathcal{D}}$ for all $t > 0$ (and $\|T\|_{B(Y)} \leq \|T\|_{B(X)}$), and there exists a uniquely determined map*

$$S(\cdot) : [0, \infty) \to B(Y, X)$$

with the following properties:

1. $S(0) = 0$;
2. $\|S(t)x - S(u)x\| \leq |t - u| \|x\|_Y \quad (t, u \geq 0; x \in Y)$;
3. $F(t)x = t \int_0^\infty e^{-tu} S(u)x \, du \quad (t > 0; x \in Y)$.

Moreover, the pair (Y,S) is "maximal-unique" in the usual sense.

Proof. The basic properties of Y are verified in precisely the same way as the corresponding properties of the Laplace–Stieltjes space W.

Denote by Lip_o the space of all complex functions f on $[0, \infty)$ such that $f(0) = 0$ and

$$|f(t) - f(u)| \leq M|t - u| \quad (t, u \geq 0).$$

The smallest possible constant M above is called the Lipshitz constant for the *Lipshitz function* f.

The remainder of the proof depends on the following

Lemma. *Let $h : \mathbb{R}^+ \to \mathbb{C}$ and $K > 0$ be given. Then there exists $f \in Lip_o$ with Lipshitz constant $\leq K$ such that $h(t)/t$ is the Laplace transform of f on $\mathbb{R}^+$ if and only if h is continuous and*

$$\left| \int_0^\infty \phi(t) h(t) \, dt \right| \leq K \, \|\mathcal{L}\phi\|_1 \tag{1}$$

for all $\phi \in C_c^\infty$.

Proof of Lemma. If $h(t)/t = (\mathcal{L}f)(t)$ for all $t > 0$, where $f \in Lip_o$ has Lipshitz constant $\leq K$, then h is clearly continuous, and f is locally absolutely continuous, its derivative (which exists a.e.) satisfies $\|f'\|_\infty \leq K$, and

$$f(t) = (Jf')(t) := \int_0^t f'(s) \, ds \quad (t \geq 0).$$

For any $\phi \in C_c^\infty$, integration by parts and Fubini's theorem give

$$\begin{aligned} \int_0^\infty \phi(t) h(t) \, dt &= \int_0^\infty \int_0^\infty t e^{-tu} (Jf')(u) \, du \, \phi(t) \, dt \\ &= \int_0^\infty \int_0^\infty e^{-tu} f'(u) \, du \, \phi(t) \, dt = \int_0^\infty (\mathcal{L}\phi)(u) f'(u) \, du. \end{aligned}$$

Thus

$$\left|\int_0^\infty \phi(t)h(t)\,dt\right| \le \|f'\|_\infty \|\mathcal{L}\phi\|_1 \le K\,\|\mathcal{L}\phi\|_1$$

for all $\phi \in C_c^\infty$.

Conversely, suppose h is continuous on $\mathbb{R}^+$ and satisfies (1).

For any $\phi \in C_c^\infty := C_c^\infty(\mathbb{R}^+)$,

$$\|\mathcal{L}\phi\|_1 \le \int_0^\infty \int_0^\infty te^{-tu}\,du|\phi(t)|\,dt/t = \|\phi\|_{L^1(\mathbb{R}^+,\,dt/t)}.$$

That is, the operator

$$\mathcal{L} : L^1(dt/t) := L^1(\mathbb{R}^+,\,dt/t) \to L^1(dt) := L^1(\mathbb{R}^+,\,dt) \tag{2}$$

is a contraction.

We identify $[L^1(dt/t)]^*$ with the space of all complex measurable functions h on $\mathbb{R}^+$ such that $th(t) \in L^\infty := L^\infty(\mathbb{R}^+,\,dt)$, normed by the essential supremum norm of $th(t)$, with the duality given by

$$\langle \phi, h\rangle = \int_0^\infty \phi(t)h(t)\,dt = \int_0^\infty \phi(t)[th(t)](dt/t).$$

By Fubini's theorem, for all $\phi \in L^1(dt/t)$ and $\psi \in L^\infty$,

$$\langle \mathcal{L}\phi, \psi\rangle = \int_0^\infty (\mathcal{L}\phi)(s)\psi(s)\,ds = \int_0^\infty (\mathcal{L}\psi)(t)\phi(t)\,dt.$$

(The use of Fubini's theorem is justified because

$$\int_0^\infty \int_0^\infty e^{-st}|\phi(t)|\,|\psi(s)|\,dt\,ds \le \|\psi\|_\infty \|\phi\|_{L^1(dt/t)} < \infty.)$$

This shows that the operator $\mathcal{L}$ defined in (2) has the adjoint

$$\mathcal{L}^* = \mathcal{L} : L^\infty(dt) \to [L^1(dt/t)]^*.$$

By (1) for h,

$$\left|\int_0^\infty [th(t)][\phi(t)/t]\,dt\right| \le K\,\|\mathcal{L}\phi\|_1 \le K\,\|\phi\|_{L^1(dt/t)}$$

for all $\phi \in C_c^\infty$, and therefore $\|th(t)\|_\infty \le K$. This means that $h \in [L^1(dt/t)]^*$, and (1) is precisely Condition (1) in Lemma 2.16 for the operator $T = \mathcal{L}$. There exists therefore $\psi \in L^\infty(dt)$ with $\|\psi\|_\infty \le K$, such that $h = \mathcal{L}^*\psi (= \mathcal{L}\psi)$ (everywhere on $\mathbb{R}^+$, by continuity of both sides). Now $f := J\psi \in Lip_o$, with Lipshitz constant $\|\psi\|_\infty \le K$, and an integration by parts shows that $h(t) = t\int_0^\infty e^{-ts} f(s)\,ds$. □

Proof of Theorem 2.23. Fix $x \in Y$ and $x^* \in X^*$. The function $h := x^*F(\cdot)x$ satisfies the criterion in the lemma with $K = \|x\|_Y \|x^*\|$. There exists therefore a unique function $f = f(\cdot; x, x^*) \in Lip_o$, such that

$$x^*F(t)x = t\int_0^\infty e^{-tu} f(u; x, x^*)\, du \quad (t > 0) \tag{3}$$

and

$$|f(t; x, x^*) - f(u; x, x^*)| \le |t - u|\, \|x\|_Y \|x^*\| \quad (t, u \ge 0). \tag{4}$$

In particular (for $u = 0$),

$$|f(t; x, x^*)| \le |t|\, \|x\|_Y \|x^*\| \quad (t \ge 0; x \in Y, x^* \in X^*).$$

The uniqueness of the representation (3) implies that $f(t; \cdot, \cdot)$ is a bounded bilinear form (for each fixed t), and there exists therefore a uniquely determined operator $S(t) \in B(Y, X^{**})$ such that

$$f(t; x, x^*) = [S(t)x](x^*) \quad (x \in Y,\, x^* \in X^*,\, t \ge 0), \tag{5}$$

and by (4),

$$\|S(t)x - S(u)x\|_{X^{**}} \le |t - u|\, \|x\|_Y \tag{6}$$

for all $t, u \ge 0$ and $x \in Y$.

For $t = 0$, the left side of (5) vanishes for all x, x^*, and therefore $S(0) = 0$.

By (6), the integral $\int_0^\infty e^{-tu} S(u)x\, du$ (with $t > 0$ and $x \in Y$) converges strongly in X^{**}, and we may then rewrite (3) in the form

$$\kappa[F(t)x] = t\int_0^\infty e^{-tu} S(u)x\, du \quad (x \in Y,\, t > 0), \tag{7}$$

where κ denotes the canonical embedding of X into X^{**}.

Let π denote the canonical homomorphism

$$\pi : X^{**} \to X^{**}/\kappa X.$$

Since π is continuous and $\pi\kappa = 0$, we obtain

$$0 = \pi\kappa[F(t)x] = t\int_0^\infty e^{-tu} \pi[S(u)x]\, du \quad (t > 0).$$

The uniqueness of the Laplace transforms implies that $\pi[S(u)x] = 0$, i.e., $S(u)x \in \kappa X$ for all $u \ge 0$ and $x \in Y$. Identifying as usual κX with X, we may restate (6) and (7) as Statements 2 and 3 of the theorem.

The maximal-uniqueness is an immediate consequence of the necessity part of the lemma and the uniqueness of the Laplace transform. □

Note that Statements 1 and 2 in Theorem 2.23 mean that $S(\cdot)$ is of class $Lip_{0,1}$ as a $B(Y,X)$-valued function (where the second subscript 1 indicates that the Lipshitz constant is ≤ 1), that is,

$$\|S(t) - S(u)\|_{B(Y,X)} \le |t-u| \quad (t,u \ge 0). \tag{8}$$

In particular, the Laplace transform $\mathcal{L}S$ exists in the $B(Y,X)$-norm on $(0,\infty)$, and by Statement 3 of the theorem, $F(\cdot)$ is $B(Y,X)$-valued and $F(t) = t(\mathcal{L}S)(t)$ for all $t > 0$. We express this relation between $F(\cdot)$ and $S(\cdot)$ by saying that $F(\cdot)$ is *the integrated Laplace transform of the* $B(Y,X)$*-valued* $Lip_{0,1}$*-function* $S(\cdot)$.

Corollary 2.24. *Let* $F(\cdot)$ *be a family of closed operators on* $(0,\infty)$*, operating on the arbitrary Banach space* X*, and let* $Y := (Y, \|\cdot\|_Y)$ *be its integrated Laplace space. Then the following statements are equivalent.*

(i) $Y = X$.
(ii) $K := \sup_{\|x\|=1} \|x\|_Y < \infty$.
(iii) $F(\cdot)$ *is the integrated Laplace transform of a* $B(X)$*-valued* $Lip_{0,K}$*-function.*

(As before, the second subscript K of *Lip* indicates that the Lipschitz constant is $\le K$.)

B.4 Integrated Semigroups

In view of the characterization of semigroup generators given in Lemma 5 in the proof of Theorem 1.38, it is interesting to consider (in the context and notation of the preceding subsection) the special family $F(t) = R(t;A)$, for a given operator A. This will give us the concept of an *integrated semigroup*, and more generally, the concept of an n*-times integrated semigroup.*

Definition 2.25. *The operator* A *on the Banach space* X*, with* $(0,\infty) \subset \rho(A)$*, is said to generate an integrated semigroup of bounded type* $\le K$ *if* $R(\cdot;A)$ *is the integrated Laplace transform of a* $B(X)$*-valued* $Lip_{0,K}$*-function* $S(\cdot)$ *on* $[0,\infty)$.

The (uniquely determined) function $S(\cdot)$ *is called the integrated semigroup generated by* A.

By Corollary 2.24, we have

Corollary 2.26. *An operator* A *with* $(0,\infty) \subset \rho(A)$*, acting in a Banach space* X*, is the generator of an integrated semigroup of bounded type* $\le K$ *if and only if*

$$\left\| \int_0^\infty \phi(t) R(t;A)\, dt \right\| \le K\, \|\mathcal{L}\phi\|_1$$

for all $\phi \in C_c^\infty(\mathbb{R}^+)$.

(Recall that $\|\cdot\|_1$ denotes the $L^1(\mathbb{R}^+, dt)$-norm.)

The more general case where $(a,\infty) \subset \rho(A)$ for some $a \geq 0$ and $S(\cdot)$ is of exponential type $\leq a$ is easily reduced to the case above by translation. We omit the details.

If $n \in \mathbb{N}$ is given, the operator A (with $(0,\infty) \subset \rho(A)$) generates an *$n$-times integrated semigroup of bounded type $\leq K$* if $R(t;A) = t^n(\mathcal{L}S)(t)$ on $(0,\infty)$, for S as in Definition 2.25, i.e., if $t^{-(n-1)}R(t;A)$ is the integrated Laplace transform of such a function S. These objects have been studied recently, and have been found to be useful in the analysis of the Abstract Cauchy Problem. We refer to the bibliography for additional information. We state only the following immediate consequence of Corollary 2.24 and of the above observation.

Corollary 2.27. *The operator A with $(0,\infty) \subset \rho(A)$ generates an n-times integrated semigroup of bounded type $\leq K$ if and only if*

$$\left\| \int_0^\infty \phi(t) R(t;A) \frac{dt}{t^{n-1}} \right\| \leq K \, \|\mathcal{L}\phi\|_1$$

for all $\phi \in C_c^\infty(\mathbb{R}^+)$.

There is a simple relation between exponentially bounded n-times integrated semigroups and pre-semigroups. In order to simplify the statement, we consider only the case $n = 1$ and we assume that $[0,\infty) \subset \rho(A)$ (making a translation as needed).

Theorem 2.28. *Let A be such that $[0,\infty) \subset \rho(A)$. Then A generates the integrated semigroup $S(\cdot)$ (of bounded type $\leq K$) if and only if it generates the pre-semigroup $A^{-1} + S(\cdot)$ (of class $Lip_{A^{-1},K}$).*

(The first subscript of *Lip* indicates the value at 0 of the functions in the class.)

Proof. The characterization in Lemma 5 (in the proof of Theorem 1.38) carries over to pre-semigroups in the following form:

The operator A (with $[0,\infty) \subset \rho(A)$) generates the pre-semigroup $W(\cdot)$ of class $Lip_{A^{-1}}$ if and only if

$$A^{-1}R(t;A)x = \int_0^\infty e^{-tu} W(u)x \, du$$

for all $t > 0, x \in X$.

Writing briefly $\mathcal{L}W$ for the above Laplace transform (understood in the strong operator topology), since $\mathcal{L}A^{-1} = t^{-1}A^{-1}$, the above condition is equivalent to

$$A^{-1}[-t^{-1} + R(t;A)] = \mathcal{L}(-A^{-1} + W)(t) \quad (t > 0),$$

that is,

$$A^{-1}[-I + tR(t; A)] = t\mathcal{L}(-A^{-1} + W)(t),$$

i.e.,

$$R(t; A) = t(\mathcal{L}S)(t) \quad (t > 0),$$

where $S := -A^{-1} + W$. □

Remark. The construction of the semi-simplicity space for A presented in Section A of Part II depends on the assumption that the resolvent set of A contains at least a half-line. In the following examples, we consider two densely defined closed operators with *empty resolvent set*, whose domains, renormed with a larger norm (the graph norm!), are Banach subspaces on which they are spectral of scalar type. We use the notation of the examples at the end of Section F, Part I.

Consider $\mathcal{F}$ with maximal domain on $L^1 := L^1(\mathbb{R}^n)$

$$D(\mathcal{F}) = \{f \in L^1;\ \mathcal{F}f \in L^1\}.$$

By Proposition 6.1.1 in [DLK1], $\mathcal{F}$ is closed, densely defined, and has empty resolvent set. Let $[D(\mathcal{F})]$ denote the Banach space $D(\mathcal{F})$ with the graph norm

$$\|f\|_F := \|f\|_1 + \|\mathcal{F}f\|_1 \quad f \in D(\mathcal{F}).$$

Then $\mathcal{F}$ is an isometry of $[D(\mathcal{F})]$ onto itself (with the inverse $\mathcal{F}^{-1} = J\mathcal{F}$, where $(Jf)(x) = f(-x)$), and is a scalar-type spectral operator on $[D(\mathcal{F})]$, with spectrum $\{i^k;\ k = 0, \ldots, 3\}$, and with the spectral integral representation

$$\mathcal{F} = \int_{\sigma(\mathcal{F})} \lambda\, P(d\lambda) = \sum_{k=0}^{3} i^k P_k,$$

where the spectral projections P_k of the resolution of the identity for $\mathcal{F}$ (on $[D(\mathcal{F})]$) are given by

$$P_k = (1/4) \sum_{j=0}^{3} (-i)^{jk} \mathcal{F}^j.$$

(Cf. Proposition 6.1.5 in [DLK1].)

Similarly, the Hilbert transform H with maximal domain in $L^1 := L^1(\mathbb{R})$

$$D(H) = \{f \in L^1;\ Hf \in L^1\},$$

is closable and has empty resolvent set. Its closure $\overline{H}$ is an isometry of the Banach subspace $[D(\overline{H})]$ onto itself, with the inverse $-\overline{H}$, and is a scalar-type spectral operator (on $[D(\overline{H})]$!), whose spectral integral representation is

$$\overline{H} = iP_i + (-i)P_{-i},$$

where the spectral *projections* of the resolution of the identity are given by

$$P_i := (1/2)(I - i\overline{H}); \quad P_{-i} := (1/2)(I + i\overline{H}).$$

(Cf. Propositions 6.2.1 and 6.2.2 in [DLK1].)

C

Families of Unbounded Symmetric Operators

In this section, Stone's integral representation theorem is generalized to certain "local" families of *unbounded* symmetric operators. We present first the Klein–Landau solution for local symmetric *semigroups*, based on a theorem of Widder on the characterization of Laplace transforms of positive measures on $\mathbb{R}$. This result has been extended by D.S. Shucker [Sh] to local symmetric semigroups on $\mathbb{R}^n$. A nice application to the Klein–Landau theorem is a transparent proof of Nelson's "Analytic Vectors Theorem," which is presented in the next subsection. We then consider local symmetric cosine families $C(\cdot)$ of (unbounded) operators. In the "bounded below" case, the method of Frohlich [Fr] for local semigroups is adapted to obtain a positive selfadjoint operator A such that

$$C(t)x = \cosh(tA^{1/2})x := \int_0^\infty \cosh(ts^{1/2})E(ds)x$$

"locally" for all x in a dense linear manifold D in X. This spectral integral local representation is then generalized to an arbitrary local symmetric cosine family; we prove the existence of a selfadjoint operator A such that

$$C(t)x = \cosh[t(A^+)^{1/2}]x + \cos[t(A^-)^{1/2}]x$$

locally for all $x \in D$, with the obvious integral representation meaning in terms of the resolution of the identity E for A.

C.1 Local Symmetric Semigroups

Let $\Delta = [0, c](c > 0)$, and let $\{T(t); t \in \Delta\}$ be a family of *unbounded* operators acting in a *Hilbert* space X, with $T(0) = I$ and $D(T(t)) := D_t(t \in \Delta)$.

Condition 1 (The domains condition). $D_s \subset D_t$ for $s \geq t$, and the linear manifold

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Progress in Mathematics 281, DOI 10.1007/978-0-8176-4932-6_10,

$$D := \bigcup \{D_t; 0 < t \leq c\}$$

is *dense* in X.

Condition 2 (The "semigroup condition," with domains taken into account). For $t, s, t+s \in \Delta$, $T(s)D_{t+s} \subset D_t$, and

$$T(t)T(s) = T(t+s) \quad \text{on} \quad D_{t+s}.$$

Condition 3 (The weak continuity condition). For each $0 < s \in \Delta$ and $x \in D_s$, $(T(\cdot)x, x)$ is continuous on $[0, s]$.

A family $T(\cdot)$ of operators satisfying the above conditions will be called a *local semigroup* (on Δ). It is *symmetric* if each $T(t)$ is a symmetric operator $(t \in \Delta)$, that is,

$$(T(t)x, y) = (x, T(t)y) \quad (x, y \in D_t, t \in \Delta).$$

Theorem 2.29 (Frohlich, Klein, and Landau). *Let $T(\cdot)$ be a symmetric local semigroup on Δ. Then there exists a unique selfadjoint operator H such that for all $0 < s \in \Delta$, $D_s \subset D(e^{-tH})$ and $T(t) = e^{-tH}$ on D_s for all $t \in [0, s/2]$.*

Proof. Fix $0 < s \in \Delta$ and $x \in D_s$, and consider the *continuous* function

$$f(t) := (T(t)x, x) \quad (t \in [0, s]).$$

For any $n \in \mathbb{N}$, if $t_1, \ldots, t_n \in [0, s]$ are such that $t_i + t_j \in [0, s]$, then $x \in D_{t_i}, T(t_i)x \in D_{s-t_i} \subset D_{t_j}$ (since $s - t_i \geq t_j$), and $T(t_j)T(t_i)x = T(t_i + t_j)x$. Therefore, by the symmetry hypothesis,

$$f(t_i + t_j) = (T(t_i)x, T(t_j)x).$$

If now $c_1, \ldots, c_n \in \mathbb{C}$, then

$$\sum_{i,j=1}^{n} c_i \overline{c_j} f(t_i + t_j) = \sum_{i,j} c_i \overline{c_j} (T(t_i)x, T(t_j)x) = \left\| \sum_i c_i T(t_i)x \right\|^2 \geq 0.$$

By a theorem of Widder [W1], this positivity property of the continuous function f on $[0, s]$ implies the existence of a unique regular positive Borel measure $\alpha = \alpha(\cdot; x)$ on $\mathbb{R}$ such that for each $t \in [0, s]$, $h_t(u) := e^{-tu} \in L^1(\alpha) := L^1(\mathbb{R}, \alpha)$ and

$$f(t) = \int_{\mathbb{R}} e^{-tu} \alpha(du). \tag{1}$$

For $n, m \in \mathbb{N}$, $c_1, \ldots, c_n; d_1, \ldots, d_m$ complex, and $s_1, \ldots, s_n; t_1, \ldots, t_m$ real such that $s_i + t_j \in [0, s]$, the preceding calculation shows that

$$\begin{aligned}\left(\sum_{i=1}^{n} c_i T(s_i)x, \sum_{j=1}^{m} d_j T(t_j)x\right) &= \sum_{i,j} c_i \overline{d_j} (T(s_i)x, T(t_j)x) \\ &= \sum_{i,j} c_i \overline{d_j} f(s_i + t_j) \\ &= \sum_{i,j} c_i \overline{d_j} \int_{\mathbb{R}} e^{-(s_i+t_j)u} \alpha(du) \\ &= \int_{\mathbb{R}} \left(\sum_i c_i e^{-s_i u}\right) \overline{\sum_j d_j e^{-t_j u}} \alpha(du). \qquad (2)\end{aligned}$$

Let Y be the closed linear span of $\{T(t)x; t \in [0, s/2]\}$, and let $U(T(t)x) := e^{-tu} (\in L^2(\alpha) := L^2(\mathbb{R}, \alpha)$, since $e^{-2tu} \in L^1(\alpha)$ for $t \in [0, s/2])$. If $g \in L^2(\alpha)$ is orthogonal to all the functions e^{-tu} with $t \in [0, s/2]$, then the function $G(z) := \int_{\mathbb{R}} e^{-zu} \overline{g(u)} \alpha(du)$, which is analytic in the strip

$$S := \{z \in \mathbb{C}; \Re z \in (0, s/2)\}$$

and continuous in its closure $\overline{S}$, must vanish identically. Hence

$$\int_{\mathbb{R}} e^{-iru} \overline{g(u)} \alpha(du) = 0$$

for all real r. By the uniqueness property of the Fourier–Stieltjes transform, it follows that $\overline{g(u)} \alpha(du) = 0$, and therefore $\int_{\mathbb{R}} g\overline{g}\, \alpha(du) = 0$, i.e., g is the zero element of $L^2(\alpha)$. Thus $\{e^{-tu}; t \in [0, s/2]\}$ is fundamental in $L^2(\alpha)$, and it follows from (2) that U extends linearly to a unitary operator from Y onto $L^2(\alpha)$.

For each $z \in \overline{S}$, the function $h_z(u) := e^{-zu}$ is in $L^2(\alpha)$ (by Widder's theorem). Moreover, the $L^2(\alpha)$-valued function $z \to h_z$ is continuous in $\overline{S}$. Indeed, let $z, w \in \overline{S}$, and denote $\Re z = t, \Re w = r$. Then

$$\begin{aligned}|e^{-zu} - e^{-wu}|^2 &\le (e^{-tu} + e^{-ru})^2 \\ &= e^{-2tu} + e^{-2ru} + 2e^{-(t+r)u} \le 4\phi(u),\end{aligned}$$

where ϕ is the $L^1(\alpha)$-function defined by $\phi(u) = 0$ for $u \ge 0$ and $\phi(u) = e^{-su}$ for $u < 0$ (recall that $2t, 2r, t + r \le s$ and Widder's theorem!). It then follows by dominated convergence that

$$\|h_z - h_w\|^2_{L^2(\alpha)} \to 0$$

when $w \to z$.

For each $g \in L^2(\alpha)$, $(h_z, g) = \int_{\mathbb{R}} e^{-zu} \overline{g(u)} \alpha(du)$ is the Laplace–Stieltjes transform of the measure $\overline{g}\, d\alpha$; it converges absolutely in $\overline{S}$ since $e^{-tu} \in L^2(\alpha)$

for $t \in [0, s/2]$, and is therefore analytic in S. Hence the $L^2(\alpha)$-valued function h_z is analytic in S.

Define

$$x(z) := U^{-1}h_z (\in Y) \quad (z \in \overline{S}). \tag{3}$$

Then, as a Y-valued function, $x(\cdot)$ is continuous in $\overline{S}$, analytic in S, and

$$x(t) = T(t)x \quad (t \in [0, s/2]). \tag{4}$$

Now, given $x \in D$, there exists $0 < s \in \Delta$ such that $x \in D_s$. Let $x(\cdot)$ be the function constructed above in the corresponding strip $\overline{S}$, and define

$$V(r)x := x(ir) \quad (r \in \mathbb{R}). \tag{5}$$

By (4) and the analyticity of $x(\cdot)$, $V(\cdot)$ is well-defined on D. For $r, r' \in \mathbb{R}$, we have

$$\begin{aligned}(V(r)x, V(r')x) &= (x(ir), x(ir')) = (h_{ir}, h_{ir'}) \\ &= \int_{\mathbb{R}} e^{-i(r-r')u} \alpha(du). \end{aligned} \tag{6}$$

Thus, by (1),

$$\|V(r)x\|^2 = \alpha(\mathbb{R}) = f(0) = \|x\|^2 \tag{7}$$

for all $r \in \mathbb{R}$ and $x \in D$, i.e., each $V(r)$ is a norm-preserving mapping from D to X. We verify its linearity as follows. If $x, y \in D$ and $\lambda, \mu \in \mathbb{C}$, there exists $0 < s \in \Delta$ such that $x, y \in D_s$. Then by (4), for all $t \in [0, s/2]$,

$$(\lambda x + \mu y)(t) = T(t)(\lambda x + \mu y) = \lambda T(t)x + \mu T(t)y = \lambda x(t) + \mu y(t),$$

and therefore, by analyticity of $x(\cdot)$ and $y(\cdot)$ in S and their continuity in $\overline{S}$, the same relation is valid with t replaced by ir, i.e., $V(r)(\lambda x + \mu y) = \lambda V(r)x + \mu V(r)y$ for all real r.

Thus $V(r)$ is an isometric linear mapping defined on the *dense* linear manifold D. It extends uniquely as a linear isometry on X (same notation). The function $r \in \mathbb{R} \to V(r)x = x(ir)$ is continuous for each $x \in D$ (as observed above), and $V(r)$ is isometric on X; therefore $V(\cdot)x$ is continuous on $\mathbb{R}$ for *all* $x \in X$.

Let $x \in D_s$ and let $t, t' \in \Delta$ be such that $t+t' \in \Delta$. The semigroup property $T(t+t')x = T(t)T(t')x$ implies, by uniqueness of the analytic continuation onto the imaginary axis, that $V(r)V(r')x = V(r+r')x$ for all $r, r' \in \mathbb{R}$ and $x \in D_s$, hence for all $x \in X$, by density of D. Thus $V(\cdot) : \mathbb{R} \to B(X)$ is a group of operators. In particular, the isometries $V(r)$ are *onto*, i.e., $V(\cdot)$ is a (strongly continuous) *unitary* group. By Stone's theorem, we have $V(r) = e^{-irH}$ with H *selfadjoint*. Let E be the resolution of the identity for H. For all $r \in \mathbb{R}$ and $x \in D_s$, we have by (6)

$$\int_{\mathbb{R}} e^{-iru}(E(du)x,x) = (V(r)x,x) = \int_{\mathbb{R}} e^{-iru}\alpha(du;x),$$

and therefore $(E(\cdot)x,x) = \alpha(\cdot;x)$, by the uniqueness property of the Fourier–Stieltjes transform.

Since $e^{-zu} \in L^2(\alpha)$ for $z \in \overline{S}$, we have $x \in D(e^{-zH})$. The vector functions $e^{-zH}x$ and $x(z)$ are both analytic in S and continuous in $\overline{S}$; on the imaginary axis, we have $e^{-irH}x = V(r)x = x(ir)$, so that $e^{-zH}x = x(z)$ for all $z \in \overline{S}$. In particular, by (4),

$$T(t)x = e^{-tH}x \quad (t \in [0,s/2], x \in D_s). \tag{8}$$

If also H' is a selfadjoint operator satisfying (8), then the analytic continuation employed in the construction gives $e^{-irH}x = e^{-irH'}x$ for all real r and all $x \in D$, hence for all $x \in X$, and therefore $H = H'$ by the uniqueness in Stone's theorem. □

The result of Theorem 2.29 is generalized in [Sh] to n-parameter local semigroups of unbounded symmetric operators, and is shown to be equivalent to the n-dimensional version of Widder's theorem.

C.2 Nelson's Analytic Vectors Theorem

We shall apply Theorem 2.29 to give a "natural" proof of Nelson's "analytic vectors theorem."

Definition 2.30. *Let A be an (unbounded) operator on X. An analytic vector for A is a vector $x \in D^\infty(A)$ such that, for some $t > 0$ (depending on x),*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\|A^n x\| < \infty.$$

Theorem 2.31 (Nelson's Analytic Vectors Theorem). *Let A be a closed symmetric operator on a Hilbert space X, that possesses a dense set D of analytic vectors. Then A is selfadjoint.*

Proof. For $t > 0$, let

$$D_t = \left\{x \in D^\infty(A); \sum \frac{t^n}{n!}\|A^n x\| < \infty\right\}.$$

These are linear manifolds such that $D_s \subset D_t$ for $s \geq t$ and $\bigcup_{t>0} D_t = D$ is dense, by hypothesis. Define $T(0) = I$ and

$$T(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x$$

for $x \in D_t, t > 0$ (the series converges in X for such x).

If $x \in D_{t+s}$,

$$\sum_m \sum_n \frac{t^m s^n}{m!n!} \|A^{m+n}x\| = \sum_k (1/k!) \sum_{m=0}^{k} \binom{k}{m} t^m s^{k-m} \|A^k x\|$$
$$= \sum_k \frac{(t+s)^k}{k!} \|A^k x\| < \infty. \tag{1}$$

Thus $\sum_n \frac{s^n}{n!} \|A^n(A^m x)\| < \infty$, that is, $A^m x \in D_s$ for all $m = 0, 1, 2, \ldots$, and since A is closed, one verifies by induction that $T(s)x \in D(A^m)$, and

$$T(s)A^m x = A^m T(s)x = \sum_n \frac{s^n}{n!} A^{n+m} x. \tag{2}$$

Therefore

$$\sum_m \frac{t^m}{m!} \|A^m T(s)x\| = \sum_m \frac{t^m}{m!} \left\| \sum_n \frac{s^n}{n!} A^{m+n} x \right\| < \infty$$

by (1), i.e., $T(s)x \in D_t$, and by (2) and absolute convergence,

$$T(t)T(s)x = \sum_m \frac{t^m}{m!} \sum_n \frac{s^n}{n!} A^{n+m} x$$
$$= \sum_k \frac{(t+s)^k}{k!} A^k x = T(t+s)x$$

(for all $t, s \geq 0$ and $x \in D_{t+s}$).

For $s > 0$ and $x \in D_s$, we have for all $t \in [0, s]$

$$(T(t)x, x) = \sum_n \frac{t^n}{n!} (A^n x, x),$$

where the series converges absolutely; in particular, $(T(\cdot)x, x)$ is continuous on $[0, s]$.

By symmetry of A, we clearly have $(T(t)x, y) = (x, T(t)y)$ for all $x, y \in D_t$.

We conclude that $\{T(t); t \geq 0\}$ is a *symmetric local semigroup*. Let H be the *selfadjoint* operator associated with it as in Theorem 2.29. If $x \in D$, then $x \in D_s$ for some $s > 0$, and

$$e^{-tH} x = T(t)x := \sum_n \frac{t^n}{n!} A^n x \tag{3}$$

for all $t \in [0, s/2]$.

In particular $e^{-tH} x \in D_{s-t} \subset D$ (for $t \in [0, s/2]$), and it follows that D is *invariant* for the C_o-(semi)group e^{-tH}. Since D is also dense in X by hypothesis, Theorem 1.7 implies that D is a *core* for the generator H, if $D \subset D(H)$.

However, for $x \in D$, we have by (3)

$$Hx := \lim_{t \to 0+} t^{-1}[e^{-tH}x - x] = Ax,$$

i.e., $x \in D(H)$ and $Hx = Ax$. Thus indeed $D \subset D(H)$, $Hx = Ax$ for all $x \in D$, and D is a core for H. Since A is closed, we then have

$$H = \overline{H|_D} = \overline{A|_D} \subset \overline{A} = A,$$

and since A is symmetric and H is selfadjoint, it follows that

$$A \subset A^* \subset H^* = H,$$

i.e., $A = H$, so that A is indeed selfadjoint. □

If A is not assumed to be closed, the theorem applies to its closure $\overline{A}$, which is closed and symmetric, and every analytic vector for A is certainly an analytic vector for $\overline{A}$. We then have

Corollary 2.32. *Let A be a symmetric operator with a dense set of analytic vectors. Then A is essentially selfadjoint.*

C.3 Local Bounded Below Cosine Families

Semigroups of operators are associated with the ACP

$$u' = Au \quad u(0) = x.$$

The "second-order ACP"

$$u'' = Au \quad u(0) = x, \quad u'(0) = 0$$

in Banach space is associated in a similar way to so-called "cosine operator functions" (cf. [G]).

Definition 2.33. *A cosine operator function on the Banach space X is a function $C(\cdot) : \mathbb{R} \to B(X)$ such that $C(0) = I$ and "D'Alembert's identity"*

$$C(t+s) + C(t-s) = 2C(t)C(s) \quad (t, s \in \mathbb{R})$$

is satisfied.

The "local version" of Definition 2.33 is the following

Definition 2.34. *Let D be a dense linear manifold in X. A local cosine family of operators on D is a family $\{C(t); t \in \mathbb{R}\}$ of operators on the Banach space X, such that for each $x \in D$ there exists $\epsilon = \epsilon(x) > 0$ so that*

(i) $x \in D(C(t))$ *and* $C(\cdot)x$ *is strongly continuous for* $|t| < \epsilon$;
(ii) $C(0)x = x$, *and for* $|t|, |s|, |t+s|, |t-s| < \epsilon$, $C(s)x \in D(C(t))$, *and*

$$C(t+s)x + C(t-s)x = 2C(t)C(s)x.$$

A result parallel to Theorem 2.29 for *symmetric* local cosine families of operators in Hilbert space is stated below, first for the special case when all the operators $C(t)$ are *bounded below*, that is,

$$(C(t)x, x) \geq \|x\|^2 \quad (x \in D(C(t)), t \in \mathbb{R}). \tag{1}$$

Condition (1) implies in particular that all the operators $C(t)$ are *symmetric*. The general case of a *symmetric local cosine family* is dealt with in Theorem 2.37.

Since no parallel to Widder's theorem [W1] is known for the cosine transform, the proof will proceed by an adaptation of Frohlich's alternative proof of Theorem 2.29.

Theorem 2.35. *Let D be a dense linear manifold in the (complex) Hilbert space X, and let $C(\cdot)$ be a local cosine family of bounded below operators on D. Then there exists a unique positive selfadjoint operator A such that*

$$C(t)x = \cosh(tA^{1/2})x$$

for all $x \in D$ and $|t| < \epsilon(x)$.

Note that the family $\{\cosh(tA^{1/2}); t \in \mathbb{R}\}$ is a cosine family of bounded below *selfadjoint* operators that *extends* the local family $C(\cdot)$.

Proof. Since $C(t)$ is *symmetric* for each t, it is closable, and its closure $\overline{C(t)}$ clearly satisfies (i), (ii), and (1). We may then assume that $C(\cdot)$ is a local cosine family of *closed* bounded below operators on D, replacing $C(t)$ by $\overline{C(t)}$ if needed (by (i), the conclusion of the theorem remains unchanged).

Fix a sequence $\{h_n\}$ of nonnegative C^∞-functions on $\mathbb{R}$, such that $h_n(t) = 0$ for $|t| \geq 1/n$ and $\int_{\mathbb{R}} h_n(t)\,dt = 1$.

Let $x \in D$, and *fix* $n(x) > 1/[\epsilon(x)]$. Denote

$$x_n = \int_{\mathbb{R}} h_n(s)C(s)x\,ds \quad (n \geq n(x)), \tag{2}$$

where the integral is a well-defined strong integral, by (i) in Definition 2.34.

Clearly $x_n \to x$ strongly. Since D is dense by hypothesis, it follows that the set

$$D_0 := \{x_n; x \in D, n \geq n(x)\}$$

is *dense* in X.

Fix $n \geq n(x)$. If $|t| < \epsilon_n(x) := \epsilon(x) - 1/n$ (note that $1/n \leq 1/n(x) < \epsilon(x)$), Condition (ii) implies that $C(s)x \in D(C(t))$ for all s with $|s| < 1/n$.

Also $C(t)C(s)x = (1/2)[C(t+s)x + C(t-s)x]$ is strongly continuous for $|s| < 1/n$ (because $|t+s|, |t-s| < \epsilon(x)$). Since $C(t)$ is closed, it follows from Theorem 3.3.2 in [HP] that

$$x_n \in D(C(t))$$

and

$$C(t)x_n = \int_{\mathbb{R}} h_n(s)C(t)C(s)x\,ds \tag{3}$$

($|t| < \epsilon_n(x)$). Let $u > 0$ be such that $|t+u|, |t-u| < \epsilon_n(x)$ (for a given t such that $|t| < \epsilon_n(x)$). By (3)

$$\begin{aligned}
&[C(t+u) + C(t-u) - 2C(t)]x_n \\
&\quad = \int_{\mathbb{R}} h_n(s)[C(t+u) + C(t-u) - 2C(t)]C(s)x\,ds \\
&\quad = \int_{\mathbb{R}} h_n(s)C(s)[\dots]x\,ds = \int_{\mathbb{R}} h_n(s)C(s)[2C(t)C(u) - 2C(t)]x\,ds \\
&\quad = \int_{\mathbb{R}} h_n(s)C(t)[2C(s)C(u) - 2C(s)]x\,ds \\
&\quad = \int_{\mathbb{R}} h_n(s)C(t)[C(s+u) + C(s-u) - 2C(s)]x\,ds \\
&\quad = \int_{\mathbb{R}} [h_n(v-u) + h_n(v+u) - 2h_n(v)]C(t)C(v)x\,dv.
\end{aligned}$$

In the last integral, integration extends over an interval where $|t|, |v|, |t+v|$, $|t-v| < \epsilon(x)$, so that Conditions (i),(ii) imply that $C(t)C(v)x$ is strongly continuous there (as a function of v), and therefore

$$\lim_{u\to 0} u^{-2}[C(t+u) + C(t-u) - 2C(t)]x_n = \int_{\mathbb{R}} h_n''(v)C(t)C(v)x\,dv \tag{4}$$

strongly (for $|t| < \epsilon_n(x)$). Let

$$D_1 = \{C(t)x_n; x \in D, n \geq n(x), 0 \leq t < \epsilon_n(x)\}.$$

Since $D_0 \subset D_1$, D_1 is dense in X, and as before, if $y \in D_1$, there exists $\epsilon'(y) > 0$ such that $y \in D(C(t))$ for $|t| < \epsilon'(y)$. By (1),

$$2u^{-2}(C(u)y - y, y) \geq 0 \quad (|u| < \epsilon'(y)). \tag{5}$$

Writing $y = C(t)x_n$ for some $n \geq n(x)$ and some $t \in [0, \epsilon_n(x))$, we have

$$\begin{aligned}
2u^{-2}[C(u)y - y] &= u^{-2}[2C(u)C(t)x_n - 2C(t)x_n] \\
&= u^{-2}[C(t+u) + C(t-u) - 2C(t)]x_n. \qquad (6)
\end{aligned}$$

By (4), the last expression has a (strong) limit as $u \to 0$, which we denote $A_0 y$. The operator A_0 is linear on the dense domain D_1, and positive (by (5) and (6)), i.e.,

$$(A_0 y, y) \geq 0 \quad (y \in D_1).$$

Let A be the Friedrichs selfadjoint extension of A_0 (cf. Theorem XII.5.2 in [DS I–III]), and let E be its resolution of the identity. Denote $E_m = E([0, m])$ and $A_m = E_m A = \int_0^m sE(ds)$ for $m \in \mathbb{N}$. Note that A_m is a *bounded* positive (selfadjoint) operator. Let $x_n \in D_0$; for $|t| < \epsilon_n(x)$,

$$\frac{d^2}{dt^2} E_m C(t) x_n = E_m A_0 x_n = A_m x_n$$

by (4) and the definition of A_0.

From the spectral representation, $\cosh(tA_m^{1/2}) E_m x_n$ is also a solution of

$$v'' = A_m v, \quad v(0) = E_m x_n, \quad v'(0) = 0.$$

By the uniqueness of the solution, we have

$$E_m C(t) x_n = \cosh(tA_m^{1/2}) E_m x_n = \cosh(tA^{1/2}) E_m x_n$$

for $|t| < \epsilon_n(x)$.

When $m \to \infty$, $E_m C(t) x_n \to C(t) x_n$ (for each $|t| < \epsilon_n(x)$); in particular, $E_m x_n \to x_n$. Also

$$\cosh(tA^{1/2}) E_m x_n = E_m C(t) x_n \to C(t) x_n$$

(when $m \to \infty$).

Since $\cosh(tA^{1/2})$ is closed, it follows that $x_n \in D(\cosh(tA^{1/2}))$ and

$$\cosh(tA^{1/2}) x_n = C(t) x_n \tag{7}$$

for $x_n \in D_0$ and $|t| < \epsilon_n(x)$.

Now let $n \to \infty$. Then $x_n \to x$, and by (3),

$$C(t) x_n = (1/2) \int_{\mathbb{R}} h_n(s)[C(t+s) + C(t-s)]x \, ds \to C(t) x$$

for $|t| < \epsilon(x)$.

Since $\cosh(tA^{1/2})$ is closed, it follows from (7) that x is in its domain, and

$$C(t) x = \cosh(tA^{1/2}) x \tag{8}$$

for all $x \in D$ and $|t| < \epsilon(x)$.

The uniqueness of A is proved as follows. If B is also a positive selfadjoint operator satisfying the identity in the theorem, and if E and F are the resolutions of the identity for A and B, respectively, then

$$\cosh(tA^{1/2}) x = \cosh(tB^{1/2}) x \quad (x \in D, |t| < \epsilon(x)).$$

Since $|\cosh z| \leq \cosh(\Re z)$, the above identity (written in term of the corresponding spectral integrals) extends analytically to t complex in the strip $|\Re t| < \epsilon(x)$. In particular, for $t \in i\mathbb{R}$, we have $\cos(sA^{1/2})x = \cos(sB^{1/2})x$ for all $x \in D$, hence for all $x \in X$ by density (since the operators are *bounded*), and for all $s \in \mathbb{R}$. Thus

$$\int_0^\infty \cos(su^{1/2})\, E(du)x = \int_0^\infty \cos(su^{1/2})\, F(du)x$$

for all $x \in X$ and $s \in \mathbb{R}$. By the uniqueness property of the cosine transform, it follows that $E = F$, and therefore $A = B$. □

C.4 Local Symmetric Cosine Families

We consider next the general case of a local cosine family of symmetric operators, *without the bounded below hypothesis.* The spectral integral representation will involve both the hyperbolic cosine and the cosine functions.

The following notation will be used. If A is a selfadjoint operator, and E is its resolution of the identity, the *positive part* A^+ and the *negative part* A^- of A are the *positive* operators defined by

$$A^+ := \int_0^\infty uE(du); \quad A^- := -\int_{-\infty}^0 uE(du)$$

with the usual domains.

The cartesian product $X^2 := \{[x, y]; x, y \in X\}$ is considered as a Hilbert space with the inner product $([x, y], [x', y']) := (x, x') + (y, y')$.

If T is an operator on X with domain $D(T)$, we let

$$\mathbb{T}[x, y] := [Tx, -Ty] \quad ([x, y] \in D(T)^2).$$

Lemma 2.36. *If T is symmetric, then $\mathbb{T}$ has a selfadjoint extension.*

Proof. Let $J[x, y] := [y, -x]$. Then J is unitary, $J^2 = -I$ (where I is the identity operator on X^2), and $JD(\mathbb{T}) = D(\mathbb{T})$. One verifies that

$$\mathbb{T} - iI = J(\mathbb{T} + iI)J.$$

Therefore

$$[\mathrm{ran}(\mathbb{T} - iI)]^\perp = J[\mathrm{ran}(\mathbb{T} + iI)]^\perp.$$

This implies that $\mathbb{T}$ (which is obviously symmetric) has equal deficiency indices ($n_- = n_+$), and has therefore a selfadjoint extension (cf. [DS I–III], Chapter XII). □

Theorem 2.37. *Let D be a dense linear manifold in the (complex) Hilbert space X, and let $C(\cdot)$ be a local cosine family of symmetric operators on D. Then there exists a selfadjoint operator A on X such that*

$$C(t)x = \cosh[t(A^+)^{1/2}]x + \cos[t(A^-)^{1/2}]x$$

for all $x \in D$ and $|t| < \epsilon(x)$.

Proof. As in the proof of Theorem 2.35, we may assume that each $C(t)$ is closed. With notation as in the proof of that theorem, we obtain the operator A_0 defined on D_1 (up to that point, the *bounded below* hypothesis was *not* used). Since

$$(C(u)y - y, y) = (y, C(u)y - y) \quad (y \in D_1, |u| < \epsilon'(y)),$$

it follows from (4) and (6) in C.3 that A_0, with the dense domain D_1, is *symmetric.* Let $\mathbb{A}$ be a selfadjoint extension of the operator $\mathbb{A}_0$ associated with A_0 (cf. Lemma 2.36), and let E be its resolution of the identity. Consider the projections $E_m^+ := E([0,m])$ and the *bounded* positive selfadjoint operators $\mathbb{A}_m^+ := E_m^+\mathbb{A}$ for $m \in \mathbb{N}$.

For $x_n \in D_0$, let

$$\xi_{nm}^+(t) := E_m^+[C(t)x_n, 0] \quad (|t| < \epsilon_n(x)).$$

Since $[C(t)x_n, 0] \in D_1^2 = D(\mathbb{A}_0)$, $\mathbb{A}_0 \subset \mathbb{A}$, and the projection E_m^+ commutes with $\mathbb{A}$, it follows from (4) in C.3 and the definition of A_0 that

$$\begin{aligned}\frac{d^2}{dt^2}\xi_{nm}^+(t) &= E_m^+[A_0C(t)x_n, 0] = E_m^+\mathbb{A}_0[C(t)x_n, 0] \\ &= E_m^+\mathbb{A}[C(t)x_n, 0] = E_m^+\mathbb{A}E_m^+[C(t)x_n, 0] = \mathbb{A}_m^+\xi_{nm}^+(t)\end{aligned}$$

for $|t| < \epsilon_n(x)$.

One verifies easily that $[\frac{d}{dt}\xi_{nm}^+](0) = 0$. By uniqueness of the solution of the second-order ACP, we then have for all $|t| < \epsilon_n(x)$:

$$\begin{aligned}\xi_{nm}^+(t) &= \cosh[t(\mathbb{A}_m^+)^{1/2}]\xi_{nm}^+(0) \\ &= \cosh[\dots]E_m^+\xi_{nm}^+(0) = \cosh[t(\mathbb{A}^+)^{1/2}]\xi_{nm}^+(0). \qquad (1)\end{aligned}$$

When $m \to \infty$,

$$\xi_{nm}^+(t) \to E([0,\infty))[C(t)x_n, 0] := \xi_n^+(t)$$

for $|t| < \epsilon_n(x)$. In particular, $\xi_{nm}^+(0) \to \xi_n^+(0)$, and $\cosh[t(\mathbb{A}^+)^{1/2}]\xi_{nm}^+(0) \to \xi_n^+(t)$ by (1). Since $\cosh[\dots]$ is closed (it is selfadjoint!), it follows that $\xi_n^+(0)$ belongs to its domain, and

$$\cosh[t(\mathbb{A}^+)^{1/2}]\xi_n^+(0) = \xi_n^+(t) \quad (|t| < \epsilon_n(x)). \qquad (2)$$

Similarly, letting $E_m^- = E([-m.0])$, etc., we find that

$$\cosh[t(-\mathbb{A}^-)^{1/2}]\xi_n^-(0) = \xi_n^-(t) \quad (|t| < \epsilon_n(x)), \tag{2'}$$

where $\xi_n^-(t) := E((-\infty, 0])[C(t)x_n, 0]$.

Adding (2) and (2′), we obtain

$$[C(t)x_n, 0] = \cosh[t(\mathbb{A}^+)^{1/2}]\xi_n^+(0) + \cos[t(\mathbb{A}^-)^{1/2}]\xi_n^-(0) \tag{3}$$

for $|t| < \epsilon_n(x)$.

Let P denote the orthogonal projection of X^2 onto X (*identified* with its embedding in X^2 as $\{[x, 0]; x \in X\}$). Then P commutes with $\mathbb{A}_0$, hence with $\mathbb{A}$, and so $A := P\mathbb{A}$ is a selfadjoint operator on X with resolution of the identity $PE(\cdot) = E(\cdot)P$. We have

$$\begin{aligned} \cosh[t(\mathbb{A}^+)^{1/2}]\xi_n^+(0) &= \int_0^\infty \cosh(tu^{1/2})E(du)[x_n, 0] \\ &= \int_0^\infty \cosh(tu^{1/2})[E(du)P][x_n, 0] = \cosh[t(A^+)^{1/2}]x_n, \end{aligned}$$

and similarly for the second term in (3) (we have identified $[x_n, 0]$ with x_n). Thus

$$C(t)x_n = \cosh[t(A^+)^{1/2}]x_n + \cos[t(A^-)^{1/2}]x_n \tag{4}$$

for all $|t| < \epsilon_n(x)$.

Recall now that for each $x \in D$, $x_n \to x$ and $C(t)x_n \to C(t)x$ when $n \to \infty$ (for $|t| < \epsilon(x)$). Since the operator on the right-hand side of (4) is closed, it follows that x belongs to its domain and the identity in Theorem 2.37 is verified. □

The following concept corresponds to that of analytic vectors in the context of cosine families.

Definition 2.38. *A semi-analytic vector for A is a vector $x \in D^\infty(A)$ such that*

$$\sum_{n=0}^\infty \frac{t^{2n}}{(2n)!}\|A^n x\| < \infty$$

for some $t > 0$ (depending on x).

If A is a closed positive operator, we may imitate the proof of Theorem 2.31, applying Theorem 2.35 to the local cosine family of *bounded below operators*

$$C(t)x = \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!}A^n x,$$

to obtain the following result:

Theorem 2.39 (Nussbaum's Semi-Analytic Vectors Theorem). *Let A be a closed positive operator with a dense set of semi-analytic vectors. Then A is selfadjoint.*

The details of the proof are left as an exercise. Note that if A is not assumed to be closed, the conclusion is that A is essentially selfadjoint.

Part III

A Taste of Applications

Prelude

Applications of semigroups cover an immense range, in areas such as Partial Differential Equations, Probability Theory, Stochastic Processes, Mathematical Physics, etc. There are excellent books exposing these subjects, some of which are listed in the bibliography. We thought that going seriously in this direction would take us too much afield, with a great amount of needed prerequisites, with wasteful overlapping with other well-known textbooks and monographs, and with a divergence from the original purpose of this book as a sort of synthesis between semigroup theory and spectral theory. Our purpose therefore in this "part" is to give a mere glimpse at some results obtained by using methods of the theory of semigroups, chosen by two criteria: the tools of semigroup theory are essentially used in their derivation, and the author was involved in their establishment.

In Section A, the results of Section D in Part I on analytic families of semigroups are applied to analytic one-parameter families of evolution systems in the "temporally inhomogeneous" case (the case of an analytic family of semigroups is the special case of a "temporally homogeneous" system). In Section B, the results of Section F in Part I on boundary values of regular semigroups are applied to obtain results on the similarity of operators within the family $S + \zeta V$ ($\zeta \in \mathbb{C}$), when iS generates a C_o-group and $V \in B(X)$ is an "S-Volterra" operator embedded in a regular semigroup.

A

Analytic Families of Evolution Systems

Let Ω be a domain in $\mathbb{C}$. For each $t \in [0, 1]$ and $z \in \Omega$, let $A(t, z)$ be an operator with dense domain D independent of t and z, in the Banach space X. Consider the one-parameter family of Cauchy problems (depending on the parameter z)

$$\frac{dx(t,z)}{dt} = A(t,z)x(t,z); \quad x(0,z) = y, \tag{1}$$

with initial value $y \in D$ and solution $x(t, z)$ wanted in D, for all $t \in [0, 1]$ and $z \in \Omega$.

In this section, we investigate the hereditary property of *analyticity* (with respect to $z \in \Omega$) from the "coefficient operator" $A(\cdot, \cdot)$ or its resolvent, to the solution $x(\cdot, \cdot)$ of the so-called (one-parameter) family of evolution systems (1).

The case when A is independent of "time" t (the so-called "temporally homogeneous" case) corresponds through Theorem 1.2 to an analytic family of semigroups, and was studied in Section D of Part I. The present discussion of the general (or "temporally inhomogeneous") case depends in an essential way on the results for the temporally homogeneous case, and on the detailed structure of the solutions of (1) constructed by Kato and Tanabe (cf. [Y, pp. 432–443]).

A.1 Coefficients Analyticity and Solutions Analyticity

Terminology 3.1. As in Section D of Part I, we consider *either one* of the following plausible *analyticity hypotheses*:

(a) *Coefficients analyticity.* For each $t \in [0, 1]$ and $x \in D$, the vector function $A(t, \cdot)x$ is analytic in Ω.
(b) *Resolvents analyticity.* For each $t \in [0, 1]$, $z \in \Omega$, and $\lambda > 0$, the resolvent $R(\lambda; A(t, z))$ exists (as an element of $B(X)$), and is analytic with respect to the parameter z in Ω.

S. Kantorovitz, *Topics in Operator Semigroups*,
Progress in Mathematics 281, DOI 10.1007/978-0-8176-4932-6_11,

The desired *analyticity conclusion* is

(c) *Solution analyticity.* For each $y \in D$ and $z \in \Omega$, the *unique* solution $x(\cdot; z)$ of (1) is analytic with respect to the parameter z in Ω.

A.2 Kato's Conditions

We adapt below Kato's conditions for the solution of (1) to the presence of the parameter $z \in \Omega$:

Condition K_1. For all $t \in [0,1]$, $z \in \Omega$ and $\lambda \geq 0$, $R(\lambda; A(t,z))$ exists and

$$\|R(\lambda; A(t,z))\| \leq 1/\lambda \quad (\lambda > 0). \tag{2}$$

In particular (case $\lambda = 0$ of K_1), $A(s,z)^{-1}X \subset D$ for all $s \in [0,1]$ and $z \in \Omega$, and therefore $A(t,z)A(s,z)^{-1} \in B(X)$ by the Closed Graph Theorem, for all $s,t \in [0,1]$ and $z \in \Omega$. Consider the differential ratios

$$B(t,s,z) := \frac{A(t,z)A(s,z)^{-1} - I}{t-s}.$$

Condition K_2. For each $x \in X$ and $z \in \Omega$, the function $B(\cdot,\cdot,z)x$ is bounded and uniformly strongly continuous off the diagonal in $[0,1]^2$, and the strong limit

$$B(t,z)x := \lim_{k\to\infty} B(t, t-1/k, z)x$$

exists uniformly for $t \in [0,1]$.

Under Conditions K_1 and K_2, the Cauchy problem has a unique solution $x(\cdot,z)$ uniformly bounded by $\|y\|$, for each given $z \in \Omega$ (cf. [Y, pp. 432–437]).

Theorem 3.2. *Under Kato's conditions* K_1 *and* K_2*, coefficients analyticity implies solution analyticity.*

Proof. For each $t \in [0,1]$ and $z \in \Omega$, it follows from Condition K_1 and Corollary 1.18 that $A(t,z)$ generates a C_o-semigroup of contractions $T(\cdot; A(t,z))$. The common domain D is *resolvent-invariant* (cf. Definition 1.65), since for all $\lambda > 0$,

$$R(\lambda; A(t,z))D \subset Domain(A(t,z)) = D.$$

Therefore, by Theorem 1.66, the semigroup $T(s; A(t,z))$ is an analytic function of z in Ω, for each $s \geq 0$ and $t \in [0,1]$.

Define the contraction-valued functions $U_k(t,s;z)$ $(k \in \mathbb{N}; t,s \in [0,1]; z \in \Omega)$ by the identities

$$U_k(t,s;z) = T(t-s; A((i-1)/k, z))$$

for $(i-1)/k \le s \le t \le i/k, (1 \le i \le k)$; and

$$U_k(t, r; z) = U_k(t, s; z) U_k(s, r; z) \quad 0 \le r \le s \le t \le 1. \tag{3}$$

The analyticity of $T(s; A(t, \cdot))$ implies the analyticity of $U_k(t, s; \cdot)$ in Ω for each $0 \le s \le t \le 1$. Therefore, for each $x \in X$ and s, t as before, the vector functions

$$x_k(t, s; \cdot) := U_k(t, s; \cdot)x \quad (k \in \mathbb{N})$$

are analytic and uniformly bounded by $\|x\|$ in Ω. By Theorem 1 in [Y, p. 432], the strong limit

$$U(t, s; z)x := \lim_k x_k(t, s; z) \tag{4}$$

exists (uniformly for s, t as above, for each $z \in \Omega$). By Lemma 1.63, it follows that $U(t, s; \cdot)x$ is analytic in Ω for each $0 \le s \le t \le 1$. By Theorem 1 in [Y, p. 432], $x(t, z) := U(t, 0; z)y$ is the unique solution of (1) uniformly bounded by $\|y\|$, and we just observed its analyticity with respect to z in Ω. □

Remark 3.3. Since $\|x_k(t, s; z)\| \le \|x\|$, it follows from (4) that $U(t, s; z)$ is a contraction for all $t, s \in [0, 1]$ and $z \in \Omega$. As a function of t, s, $U(t, s; z)$ is strongly continuous (for each fixed z), because the limit in (4) is uniform in $t, s \in [0, 1]$ and each $x_k(\cdot, \cdot; z)$ is continuous on $[0, 1]^2$. Since $U(t, s; \cdot)x$ is analytic on Ω for all $x \in X$ (for each fixed t, s; see end of preceding proof), it follows from Theorem 3.10.1 in [HP] that the operator function $U(t, s; \cdot)$ is analytic in Ω (for each fixed t, s). By (3)

$$U(t, r; z) = U(t, s; z) U(s, r; z) \quad 0 \le r \le s \le t \le 1$$

for all $z \in \Omega$. Thus $\{U(\cdot, \cdot; z); z \in \Omega\}$ is a family of strongly continuous contraction *propagators* (the so-called *propagators family* associated with the *evolution problem* (1)), that depends *analytically* on the parameter z, $z \in \Omega$. Briefly, *under Kato's Conditions K_1 and K_2, coefficients analyticity implies propagators analyticity.*

Theorem 3.4. *Under Kato's Conditions K_1 and K_2, resolvents analyticity implies solution analyticity.*

Proof. As in the proof of Theorem 2 in [Y, p. 436], we consider the contractions (for $0 \le s \le t \le 1$ and $k \in \mathbb{N}$):

$$V_k(t, s; z) := \frac{k}{\{kt\}} R\left(\frac{k}{\{kt\}}; A([kt]/k, z) \times \prod_{j=[ks]+1}^{[kt]-1} kR(k; A(j/k, z)\right)$$
$$\times \frac{k}{1-\{ks\}} R\left(\frac{k}{1-\{ks\}}; A([ks]/k, z)\right),$$

where the first factor is understood as I when kt is an integer, $\{kt\}$ denotes the fractional part of kt, and the middle product runs in "reverse order."

The "resolvents analyticity hypothesis" implies that $V_k(t,s;\cdot)$ is analytic in Ω for each fixed t,s as above and $k \in \mathbb{N}$. Therefore, for each $y \in D$, the vector functions

$$v_k(t,\cdot) := V_k(t,0;\cdot)y$$

are analytic and uniformly bounded (by $\|y\|$) in Ω. By Theorem 2 in [Y, p. 436], the strong limit $x(t,z) := \lim_k v_k(t,z)$ exists (for each $t \in [0,1]$ and $z \in \Omega$), and is the unique solution of (1) uniformly bounded by $\|y\|$. Finally, we conclude from Lemma 1.63 that $x(t,\cdot)$ is analytic in Ω. □

A.3 Tanabe's Conditions

Another set of sufficient conditions for the existence and uniqueness of the solution of the evolution system (1) are Tanabe's conditions T_1 and T_2 below, which are adapted here to the presence of a parameter $z \in \Omega$.

Condition T_1. For each $t \in [0,1]$ and $z \in \Omega$, the resolvent set of $A(t,z)$ contains 0 and a sector S_θ with $\theta > \pi/2$, and $R(\lambda; A(t,z))$ is strongly continuous in t, uniformly in λ on compact subsets of S_θ, for each $z \in \Omega$. Furthermore, for each subdomain $\Omega_0 \subset\subset \Omega$, there exist constants M, N (depending on Ω_0) such that

$$\|R(\lambda; A(t,z))\| \le \frac{N}{|\lambda| - M}$$

for all $t \in [0,1]$, $z \in \Omega_0$, and $\lambda \in S_\theta$, $|\lambda| > M$ (with $N = 1$ for *real* $\lambda > M$).

Note that by T_1 and the fact that the operators $A(t,z)$ are closed (since their resolvent sets are nonempty) and have the common domain D, the operators $A(t,z)A(s,z)^{-1}$ are bounded operators for all $s,t \in [0,1]$ and $z \in \Omega$. The second Tanabe condition is a Lipshitz condition on $A(\cdot,z)A(s,z)^{-1}$, uniformly with respect to $s \in [0,1]$ and $z \in \Omega_0 \subset\subset \Omega$:

Condition T_2. For each subdomain $\Omega_0 \subset\subset \Omega$, there exists a constant K (depending on Ω_0) such that

$$\|A(t,z)A(s,z)^{-1} - A(r,z)A(s,z)^{-1}\| \le K\,|t-r|$$

for all $s,t,r \in [0,1]$ and $z \in \Omega_0$.

Lemma 3.5. *Under Tanabe's conditions* T_1 *and* T_2*, the* $B(X)$*-valued function*

$$F(t,s;\cdot) := A(t,\cdot)A(s,\cdot)^{-1}$$

is analytic in Ω*, for each given* $t,s \in [0,1]$.

Proof. Given $s \in [0,1]$, $z \in \Omega$, and $x \in X$, denote $y = A(s,z)^{-1}x$. Then $y \in D$, so that $A(t,w)y \to A(t,z)y$ when $w \to z$ ($w \in \Omega$), for each $t \in [0,1]$ (by the analyticity, hence the continuity, of $A(t,\cdot)y$). In particular, $A(s,w)y \to x$ when $w \to z$.

Fix a disk $\Omega_0 \subset\subset \Omega$ centered at z, and let the constant K correspond to Ω_0 as in Condition T_2. Then for all $t \in [0,1]$ and $w \in \Omega_0$, we have by Condition T_2

$$\|F(t,s;w)\| = \|F(t,s;w) - F(s,s;w) + I\| \le K\,|t-s| + 1 \le K+1. \qquad (5)$$

Therefore

$$\begin{aligned}
&\|F(t,s;z)x - F(t,s;w)x\| \\
&\quad = \|A(t,z)y - A(t,w)y + A(t,w)A(s,w)^{-1}[A(s,w)y - x]\| \\
&\quad \le \|A(t,z)y - A(t,w)y\| + (K+1)\,\|A(s,w)y - x\| \to 0
\end{aligned}$$

when $w \to z$, as remarked above.

This shows that $F(t,s;\cdot)$ is strongly continuous in Ω.

Consider next the differential ratio for $F(t,s;\cdot)$ at the given point z. With notation as before and $w \in \Omega_0$, $w \neq z$,

$$\begin{aligned}
&\|(z-w)^{-1}[F(t,s;z)x - F(t,s;w)x] - [A(t,\cdot)y]'(z) + F(t,s;z)[A(s,\cdot)y]'(z)\| \\
&\quad \le \|(z-w)^{-1}[A(t,z)y - A(t,w)y] - [A(t,\cdot)y]'(z)\| \\
&\qquad + \left\| A(t,w)\frac{A(s,z)^{-1}x - A(s,w)^{-1}x}{z-w} + F(t,s;z)[A(s,\cdot)y]'(z) \right\| \\
&\quad = I + II.
\end{aligned}$$

Clearly $I \to 0$ as $w \to z$, because $A(t,\cdot)y$ is analytic at z. We have

$$\begin{aligned}
II &= \left\| F(t,s;w)\frac{A(s,w) - A(s,z)}{z-w}y + F(t,s;z)[A(s,\cdot)y]'(z) \right\| \\
&\le \left\| F(t,s;w)\left[-\frac{A(s,w)y - A(s,z)y}{w-z} + [A(s,\cdot)y]'(z) \right] \right\| \\
&\qquad + \|[F(t,s;z) - F(t,s;w)]\,[A(s,\cdot)y]'(z)\| = III + IV.
\end{aligned}$$

By (5)

$$III \le (K+1)\,\|(w-z)^{-1}[A(s,w)y - A(s,z)y] - [A(s,\cdot)y]'(z)\| \to 0$$

as $w \to z$, by the analyticity of $A(s,\cdot)y$ at the point z. Also $IV \to 0$ as $w \to z$, by the strong continuity of $F(t,s;\cdot)$ at the point z (which we proved before). We then conclude that $F(t,s;\cdot)x$ is analytic in Ω (with the expected expression

$$[A(t,\cdot)y]'(z) - F(t,s;z)[A(s,\cdot)y]'(z)$$

for the derivative at z (where $y := A(s,z)^{-1}x$). □

Fixing $z \in \Omega$, Tanabe's theorem states that under Conditions T_1 and T_2, the Cauchy problem (1) (with arbitrary initial value $y \in X$) has a unique solution $x(\cdot, z)$ (cf. [Y, pp. 439–443]). Its dependence on z is now considered.

Theorem 3.6. *Under Tanabe's conditions T_1 and T_2, coefficients analyticity implies solution analyticity.*

Proof. The following elementary lemma is used several times in the proof.

Lemma A. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $h : [a, b] \times \Omega \to X$ be a bounded function, continuous in $t \in [a, b]$ for each $z \in \Omega$, and analytic in $z \in \Omega$ for each $t \in [a, b]$. Then $u(z) := \int_a^b h(t, z)\, dt$ is analytic in Ω.*

Proof of Lemma A. The Riemann sums

$$u_n(z) := \sum_{k=1}^{n} h(t_k, z)(t_k - t_{k-1})$$

for equipartitions $a = t_0 < t_1 < \cdots < t_n = b$ are analytic and uniformly bounded in Ω, and $u_n(z) \to u(z)$ strongly, for each $z \in \Omega$. By Lemma 1.63, u is analytic in Ω. □

We proceed now with the proof of Theorem 3.6. Condition T_1 implies that $A(s, z)$ generates a *holomorphic* semigroup $T(\cdot; A(s, z))$ (for each $(s, x) \in [0, 1] \times \Omega$). By Theorem 1.66, $T(t; A(s; \cdot))$ is analytic in Ω, for each $s \in [0, 1]$ and $t \geq 0$. For $s \in [0, 1]$ and $z \in \Omega$ *fixed*, represent the (strong) derivative $T(\cdot; A(s, z))' = A(s, z)T(\cdot; A(s, z))$ of the holomorphic semigroup $T(\cdot; A(s, z))$ by Cauchy's formula for the derivative. By the proof of Lemma 1 in [Y, p. 440], $\|T(t; A(s, z))\| \leq C$. It follows from Lemma A that $A(s, z)T(t; A(s, z))$ is analytic in z in Ω, for each fixed t, s. By Lemma 3.5, if $0 \leq s < t \leq 1$, then

$$\begin{aligned} R_1(t, s; z) &:= [A(t, z) - A(s, z)]\, T(t - s; A(s, z)) \\ &= F(t, s; z)\, [A(s, z)T(t - s; A(s, z))] - A(s, z)T(t - s; A(s, z)) \end{aligned} \quad (6)$$

is analytic in z in Ω.

As in [Y, p. 439], extend the definition (6) of $R_1(t, s; z)$ by setting

$$R_1(t, s; z) = 0 \text{ for } s \geq t \quad (t, s \in [0, 1]),$$

and define R_m inductively by

$$R_m(t, s; z)x = \int_s^t R_1(t, r; z) R_{m-1}(r, s; z)x\, dr. \quad (7)$$

Proceeding inductively, suppose that R_{m-1} is analytic in z in Ω for some $m \geq 2$. From the proof of Lemma 1 in [Y, p. 440], the integrand in (7) is continuous in r in the fixed interval $[s, t]$, $0 \leq s < t \leq 1$, for each $z \in \Omega$; it is uniformly bounded for $r \in [s,t]$ and $z \in \Omega_0 \subset\subset \Omega$ by $(KC)^m \|x\|/((m-1)!)$ (where the constants depend on Ω_0); and finally, by the induction hypothesis and the analyticity of R_1 which we observed above, the integrand in (7) is analytic in z in Ω_0. By Lemma A, it follows that $R_m(t,s;\cdot)$ is analytic in any subdomain $\Omega_0 \subset\subset \Omega$, hence in Ω, for each fixed $s < t$ (and trivially for $s \geq t$).

The estimate $\|R_m\| \leq (KC)^m/((m-1)!)$ in each subdomain $\Omega_0 \subset\subset \Omega$ implies that the series

$$R(t,s;z) := \sum_m R_m(t,s;z) \tag{8}$$

converges in $B(X)$ uniformly in $[0,1]^2 \times \Omega_0$. It follows that the function

$$T(t-r;A(r,z))R(r,s;z) \quad r \in [s,t],\ z \in \Omega_0$$

(with $s < t$ fixed) satisfies the hypothesis of Lemma A. We then conclude that the function

$$U(t,s;z) := T(t-s;A(s,z)) + \int_s^t T(t-r;A(r,z))\,R(r,s;z)\,dr \tag{9}$$

is analytic in z in each subdomain $\Omega_0 \subset\subset \Omega$ (hence in Ω), for each $s < t$ in $[0,1]$.

By Tanabe's theorem (cf. [Y, p. 439]), the unique solution of the Cauchy problem (1) (under Tanabe's conditions T_1 and T_2, with $y \in X$ arbitrary) is given by $x(t,z) = U(t,0;z)y$, and is therefore analytic with respect to z in Ω, for each $t \in [0,1]$. □

Note again that the proof above establishes the analyticity of the propagator $U(t,s;z)$, $0 \leq s < t \leq 1$, with respect to the parameter z in Ω. Briefly, *under Tanabe's conditions T_1 and T_2, coefficients analyticity implies propagators analyticity.*

B

Similarity

B.1 Overview

In this section, we shall be concerned with the existence of a similarity relation within certain families of (unbounded) operators. Two (generally unbounded) operators A, B are said to be similar if there exists a bounded nonsingular operator Q such that $B = Q^{-1}AQ$ (with equality of domains!). In [K18] and [K19], the classical Volterra operator $V : f(x) \to \int_0^x f(t)\,dt$ and the multiplication operator $f(x) \to xf(x)$, acting on $L^p(0,1)$ $(1 < p < \infty)$ were shown to possess the property that $S + \zeta V$ is similar to $S + \omega V$ $(\zeta, \omega \in \mathbb{C})$ if and only if $\Re\zeta = \Re\omega$). It turned out later (cf. [K20]) that a key to the result is the so-called "Volterra commutation relation" $[S, V] = V^2$, trivially satisfied by the special pair (S, V) defined above. If the above operators are acting in $L^p(0, \infty)$ as unbounded operators with maximal domains, the condition $\Re\zeta = \Re\omega$ is still sufficient for the similarity of $S + \zeta V$ and $S + \omega V$ (cf. [HK] and [K4]), but its necessity is unknown. In [KH4], [KPe], and [VK], abstract settings were found for the validity of similarity results of the above type with S not necessarily bounded (but with V bounded): iS generates a bounded C_o-group $S(\cdot)$, V leaves $D(S)$ invariant, satisfies the Volterra commutation relation in the form $[S, V] \subset V^2$, and can be embedded as $V = V(1)$ in a suitable regular semigroup $V(\cdot)$. In particular, under these hypotheses, $S+\zeta V$ is similar to S if and only if $\Re\zeta = 0$. For example, $S - V$ is *not* similar to S. However, we show in the last subsection that, without any assumption on the group $S(\cdot)$ and on the operator V (besides the Volterra commutation relation between S and V), the perturbations $(S - V) + P$ of $S - V$ are similar to S for all P in the similarity suborbit of V given by $\{S(-t)VS(t);\ t \in \mathbb{R}\}$.

B.2 Similarity Within the Family $S + \zeta V$

In this section, the operator iS with domain $D(S)$ in the Banach space X is the generator of a C_o-group $S(\cdot)$, and $V \in B(X)$ satisfies the "Volterra

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relation" with S, that is,

$$VD(S) \subset D(S); \quad [S,V] \subset V^2. \tag{1}$$

The domain of the Lie product $[S,V] := SV - VS$ is clearly $D(S)$. By Theorem 1.38, for any $\zeta \in \mathbb{C}$, the operator

$$T_\zeta := S + \zeta V \tag{2}$$

(with domain $D(S)$) is such that iT_ζ generates a C_o-group, which we denote by $T_\zeta(\cdot)$. We first list a number of properties of an operator $V \in B(X)$ that are equivalent to the Volterra relation.

Lemma 3.7. *The following statements are equivalent for $V \in B(X)$:*

(a) *V satisfies the Volterra relation with S.*
(b) $[R(\lambda;S),V] = R(\lambda;S)\,V^2\,R(\lambda;S) \quad (\lambda \in \rho(S))$.
(c) $[R(\lambda;S),V] = V\,R(\lambda;S)^2 V$ *for all complex λ outside some horizontal strip*

$$\Delta(c,c') := \{\lambda \in \mathbb{C}; -c' \le \Im\lambda \le c\},$$

$c, c' \ge 0$.
(d) $[R(\lambda;S)^n, V] = nV\,R(\lambda;S)^{n=1}V$ *for all $n \in \mathbb{N}$ and for all complex λ outside some strip $\Delta(c,c')$.*
(e) $[S(t),V] = itV\,S(t)\,V$ *for all $t \in \mathbb{R}$.*
(f) *Same as (e), for all t in some interval $(0,\delta)$.*

Proof. (a) *implies* (b). It follows from (a) that for all $x \in D(S)$,

$$(\lambda I - S)Vx = V(\lambda I - S)x - V^2 x \quad (\lambda \in \mathbb{C}). \tag{3}$$

For each $x \in X$ and $\lambda \in \rho(S)$, we have $R(\lambda;S)x \in D(S)$, hence $V\,R(\lambda;S)x \in D(S)$ by (a), and it follows from (3) and (a) that

$$\begin{aligned} V\,R(\lambda;S)x &= R(\lambda;S)[(\lambda I - S)V]R(\lambda;S)x \\ &= R(\lambda;S)[V(\lambda I - S) - V^2]R(\lambda;S)x \\ &= R(\lambda;S)Vx - R(\lambda;S)\,V^2 R(\lambda;S)x, \end{aligned}$$

which proves (b).

(b) *implies* (c). Since $R(\lambda;iS) = -iR(-i\lambda;S)$ (whenever either side exists), it follows from Theorem 1.39 that there exists a strip $\Delta(c,c')$ such that $R(\cdot;S)$ is a well-defined analytic $B(X)$-valued function outside the strip, and $\|R(\lambda;S)\| \to 0$ as $\Im\lambda \to \infty$. We may then choose $r > 0$ such that

$$\|V\,R(\lambda;S)\| < 1 \quad (|\Im\lambda| > r). \tag{4}$$

By (b), for all $\lambda \in \rho(S)$,

$$V\,R(\lambda; S) = R(\lambda; S)V(I - V\,R(\lambda; S)). \tag{5}$$

By (4), if $|\Im\lambda| > r$, the inverse $(I - R(\lambda; S))^{-1}$ exists in $B(X)$ and is given by the usual "geometric series"; it then follows from (5) that

$$\begin{aligned} R(\lambda; S)V &= V\,R(\lambda; S)\,(I - V\,R(\lambda; S))^{-1} \\ &= \sum_{n=1}^{\infty}(V\,R(\lambda; S))^n. \end{aligned} \tag{6}$$

The series representation in (6) implies that $R(\lambda; S)V$ commutes with $V\,R(\lambda; S)$ for $|\Im\lambda| > r$, and therefore, by analyticity outside $\Delta(c, c')$ of both sides of the relevant commutation identity, the latter is valid for all λ outside the strip, and (c) follows from (b).

(c) *implies* (d). We use induction on n. The case $n = 1$ is precisely our hypothesis (c). Assume (d) for n. By the resolvent identity (cf. Theorem 1.11),

$$\frac{d}{d\lambda}R(\lambda; S) = -R(\lambda; S)^2 \quad (\lambda \in \rho(S)),$$

where the derivative is taken in the uniform operator topology. Differentiating the identity (d) for n (our induction hypothesis!) using the chain rule, we obtain for $\lambda \notin \Delta(c, c')$

$$-n[R(\lambda; S)^{n+1}, V] = -n(n+1)V\,R(\lambda; S)^{n+2}V,$$

and (d) for $n + 1$ follows.

(d) *implies* (e). Fix $t > 0$. Since iS generates the C_o-semigroup $\{S(t);\, t \geq 0\}$, we have by Theorem 1.36

$$[S(t), V] = \lim_n \left[\left(\frac{n}{t}R(\frac{n}{t}; iS)\right)^n, V\right] = \lim_n \left[\left(\frac{n}{it}R\left(\frac{n}{it}; S\right)\right)^n, V\right],$$

in the strong operator topology (s.o.t.), as $n \to \infty$.

Hence by (d), in the s.o.t.,

$$\begin{aligned} [S(t), V] &= \lim_n (n/it)^n nV\,R(n/it; S)^{n+1}V \\ &= itV\,\lim_n \left(\frac{n}{it}R\left(\frac{n}{it}; S\right)\right)\,\left(\frac{n}{it}R\left(\frac{n}{it}; S\right)\right)^n V. \end{aligned}$$

However, the first factor in the last limit converges to the identity operator in the s.o.t. as $n \to \infty$ (cf. Lemma 1.16) and has operator norm < 2 for n sufficiently large; the second factor converges to $S(t)$ in the s.o.t. (as observed above). Hence (e) follows for $t > 0$. A similar calculation yields (e) for $t < 0$, using the corresponding formula for the C_o-semigroup $\{S(-t); t \geq 0\}$ (the case $t = 0$ is trivial).

(f) *implies* (a). Let $x \in D(S)$ and $t \in (0, \delta)$ (with $\delta > 0$ as in (f)). We have by (f)

$$\frac{S(t)\,Vx - Vx}{t} = V\frac{S(t)x - x}{t} + iV\,S(t)\,Vx \to iVSx + iV^2x$$

as $t \to 0$. Hence $Vx \in D(S)$ and $iS(Vx) = iVSx + iV^2x$. This proves (a). □

Recall the notation $T_\zeta := S + \zeta V$ ($\zeta \in \mathbb{C}$); the C_o-group generated by iT_ζ is denoted by $T_\zeta(\cdot)$.

Lemma 3.8. *Let $V \in B(X)$ satisfy the Volterra relation with S. Then $\rho(V)$ contains the imaginary axis without 0, and for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}$,*

$$T_k(t) = (I - itV)^{-k}S(t) = S(t)\,(I + itV)^k.$$

Proof. For $k \in \mathbb{N}$, denote

$$H_k(t) := S(t)(I + itV)^k \quad (t \in \mathbb{R}). \tag{7}$$

Clearly, $H_k(\cdot)$ is strongly continuous on $\mathbb{R}$ and $H_k(0) = I$.

By Lemma 3.7 (e), for all $s, t \in \mathbb{R}$,

$$\begin{aligned} H_1(s)H_1(t) &= S(s)(S(t) + isVS(t) + itS(t)V + is[S(t), V]) \\ &= S(s)S(t)(I + i(s+t)V) = H_1(s+t). \end{aligned}$$

Thus $H_1(\cdot)$ is a C_o-group.

In particular, we have

$$I + itV = S(-t)H_1(t) \quad (t \in \mathbb{R}), \tag{8}$$

hence $I + itV$ is invertible for all $t \in \mathbb{R}$, with

$$(I + itV)^{-1} = H_1(-t)S(t) \quad (t \in \mathbb{R}). \tag{9}$$

In particular, $\rho(V)$ contains the imaginary axis without 0.

Let iB be the generator of the C_o-group $H_1(\cdot)$. If $x \in D(S)$, $(I + itV)x \in D(S)$ for all $t \in \mathbb{R}$ (since $D(S)$ is V-invariant), and therefore $H_1(t)x$ is differentiable and

$$\frac{d}{dt}H_1(t)x = S(t)(iS)(I + itV)x + S(t)(iV)x.$$

In particular, the derivative at zero is $i(S + V)x$. This shows that $S + V \subset B$.

Replacing t by $-t$ in (8) and multiplying on the right by $H_1(t)$, we see that

$$S(t) = (I - itV)H_1(t) \quad (t \in \mathbb{R}). \tag{10}$$

Therefore, if $x \in D(B)$, $S(t)x = (I - itV)[H_1(t)x]$ is differentiable on $\mathbb{R}$, and

$$\frac{d}{dt}S(t)x = -iV\, H_1(t)x + (I - itV)H_1(t)(iB)x.$$

In particular, the derivative at zero is $i(B - V)x$, i.e., $D(B) \subset D(S)$ (and $B - V \subset S$). We conclude that $B = S + V := T_1$. Since the generator determines its C_o-(semi)group uniquely, it follows that $T_1(\cdot) = H_1(\cdot)$.

We now use induction on k to prove that $T_k(t) = H_k(t)$ for all $k \in \mathbb{N}$.

We proved above the case $k = 1$. Observe that V satisfies the Volterra relation with T_k, since $D(T_k) = D(S)$ is V-invariant and $[T_k, V] = [S + kV, V] = [S, V] \subset V^2$. Also $T_{k+1} = T_k + V$. Therefore, by the case $k = 1$ for the pair (T_k, V), $T_{k+1}(t) = T_k(t)(I + itV)$. Hence, by the induction hypothesis for k,

$$T_{k+1}(t) = H_k(t)(I + itV) = H_{k+1}(t),$$

as desired.

The second identity in the lemma (for $k \in \mathbb{N}$) is an easy consequence of the first: for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$T_k(t) = T_k(-t)^{-1} = [S(-t)(I - itV)^k]^{-1} = (I - itV)^{-k}S(t).$$

Consider now the operators T_{-k} and V $(k \in \mathbb{N})$. Clearly V satisfies the Volterra relation with T_{-k}, and iT_{-k} generates the C_o-group $T_{-k}(\cdot)$. By the proof above, it follows that the C_o-group generated by $i(T_{-k} + kV) = iS$, that is, the group $t \to S(t)$, coincides with the groups $t \to T_{-k}(t)(I + itV)^k$ and $t \to (I - itV)^{-k}T_{-k}(t)$. "Solving" for $T_{-k}(t)$, we get

$$T_{-k}(t) = S(t)(I + itV)^{-k} = (I - itV)^k S(t) \quad (t \in \mathbb{R},\, k \in \mathbb{N}).$$

Since the identities of the lemma are trivial for $k = 0$, the proof is complete. □

A slight extension of the identities of Lemma 3.8 is obtained by replacing S by T_ζ. As observed in the proof of the lemma, $D(T_\zeta)$ $(= D(S))$ is V-invariant and $[T_\zeta, V]$ $(= [S, V]) \subset V^2$. The said identities are then valid for the pair (T_ζ, V) instead of the pair (S, V). Since the C_o-group generated by $i(T_\zeta + kV)$ $(= iT_{\zeta+k})$ is $T_{\zeta+k}(\cdot)$, we obtain the following

Corollary 3.9. *Let $V \in B(X)$ satisfy the Volterra relation with S, where iS is the generator of the C_o-group $S(\cdot)$. For each $\zeta \in \mathbb{C}$, let $T_\zeta(\cdot)$ denote the C_o-group generated by iT_ζ, where $T_\zeta := S + \zeta V$. Then*

$$T_{\zeta+k}(t) = T_\zeta(t)(I + itV)^k = (I - itV)^{-k}T_\zeta(t)$$

for all $t \in \mathbb{R}$, $\zeta \in \mathbb{C}$, and $k \in \mathbb{Z}$.

In general, the identities in Lemma 3.8 and Corollary 3.9 do not make sense if the integer k is replaced by an arbitrary complex number ω, because $(I+itV)^{\omega}$ is not always defined. However, in case $S \in B(X)$, Lemma 3.8 does extend (with k and it replaced by arbitrary complex numbers). Of course, the groups are now ordinary exponentials.

Corollary 3.10. *Let $S, V \in B(X)$, and suppose $[S, V] = V^2$. Then V is quasi-nilpotent, and for all $z, \zeta \in \mathbb{C}$,*

$$e^{z(S+\zeta V)} = e^{zS}(I+zV)^{\zeta} = (I-zV)^{-\zeta}e^{zS}.$$

Proof. Since the claim is trivial when $V = 0$, we may assume that $\|V\| > 0$. Apply Lemma 3.8 with the group $S(t) = e^{itS}$. By the identity of the lemma with $k = 1$,

$$e^{it(S+V)} = e^{itS}(I+itV) \quad (t \in \mathbb{R}). \tag{11}$$

By (11), the $B(X)$-valued entire functions $e^{z(S+V)}$ and $e^{zS}(I+zV)$ coincide for $z \in i\mathbb{R}$, and therefore they coincide on $\mathbb{C}$. Equivalently

$$I + zV = e^{-zS}e^{z(S+V)} \quad (z \in \mathbb{C}). \tag{12}$$

In particular, $I + zV$ is invertible in $B(X)$ for all $z \in \mathbb{C}$, that is, V is quasi-nilpotent. It follows that the powers $(I+zV)^{\zeta}$ are well-defined for all $\zeta \in \mathbb{C}$ by the binomial series

$$(I+zV)^{\zeta} := \sum_{k=0}^{\infty} \binom{\zeta}{k} z^k V^k \tag{13}$$

(convergent in $B(X)$-norm).

Fix $\nu \in (\pi/2, \pi)$, $\epsilon > 0$, and $z \in \mathbb{C}$ such that $|z| < 1/\epsilon$. In the following, C, C' , etc. are positive constants. Since $\lim \|V^n\|^{1/n} = 0$, there exists k_0 such that $\|V^k\| < \epsilon^k$ for all $k > k_0$. We use the estimate

$$\left|\binom{\zeta}{k}\right| < Ce^{\nu|\zeta|},$$

holding uniformly in k (cf. [HP, p. 234]). For $k \leq k_0$, the terms in (13) have norms $\leq Ce^{\nu|\zeta|}(|z|\,\|V\|)^k$; for $k > k_0$, the terms have norms $\leq Ce^{\nu|\zeta|}(|z|\epsilon)^k$. Since $|z|\epsilon < 1$, the series in (13) is norm-majorized by a convergent series of constants for all ζ in any compact subset of $\mathbb{C}$, hence converges "absolutely" and uniformly on compacta (with respect to ζ). Therefore $(I+zV)^{\zeta}$ is an entire function of ζ, and moreover

$$\|(I+zV)^{\zeta}\| \leq Ce^{\nu|\zeta|}\left(\frac{(|z|\,\|V\|)^{k_0+1}-1}{|z|\,\|V\|-1} + \frac{1}{1-|z|\epsilon}\right) = C'e^{\nu|\zeta|}. \tag{14}$$

The function $\zeta \to e^{z(S+\zeta V)}$ is clearly entire, and has norm

$$\leq e^{|z|\,\|S+\zeta V\|} \leq e^{|z|\,\|S\|} e^{|\zeta|\,(\|V\|\,|z|)} \leq C'' e^{\nu|\zeta|}$$

if we assume that the fixed z satisfies $|z| < \nu/\|V\|$. Define

$$F_z(\zeta) := e^{-zS} e^{z(S+\zeta V)} - (I+zV)^{\zeta}.$$

By the preceding discussion, for each fixed z such that $|z| < \delta := \min(1/\epsilon, \nu/\|V\|)$, the $B(X)$-valued function $F_z(\cdot)$ is entire and of exponential type $\leq \nu < \pi$. By Lemma 3.8, $F_z(k) = 0$ for all $k \in \mathbb{Z}$. Therefore $F_z(\zeta) = 0$ for all $\zeta \in \mathbb{C}$ (cf. Theorem 3.13.7 in [HP]). Thus

$$e^{z(S+\zeta V)} = e^{zS}(I+zV)^{\zeta} \tag{15}$$

for all $\zeta \in \mathbb{C}$ and $z \in \mathbb{C}$ such that $|z| < \delta$. By the power series representation (in powers of z) of the exponentials and the binomial, both sides in (15) are entire functions of z for each fixed $\zeta \in \mathbb{C}$ (we use the fact that V is quasi-nilpotent!); since they coincide for $|z| < \delta$, they coincide for all z, and the first identity of Corollary 3.10 is established. The second follows from the first as in the proof of Lemma 3.8. □

(An alternative proof can be found in [K4].)

We shall now assume that the bounded operator V is "embedded" as $V = V(1)$ in a regular semigroup $V(\cdot)$, cf. Section F of Part I. To simplify notation, we shall use the same symbol $V(\cdot)$ for the extension to the closed halfplane $\overline{\mathbb{C}^+}$. The boundary group $V(i\cdot)$ satisfies the growth estimate $\|V(it)\| \leq C\, e^{b|t|}$, where $C \geq 1$ and $b \geq 0$. Let γ be the Norlund function of $V(\cdot)$, that is,

$$\gamma(s) := \limsup_{|t|\to\infty} |t|^{-1} \log \|V(s+it)\| \quad (s > 0). \tag{16}$$

Then $\gamma(s) \leq b$. If, for example, for any $b \in (\pi/2, \pi)$ there is a constant $C_b \geq 1$ such that $\|V(it)\| \leq C_b e^{b|t|}$, then $\gamma(s) \leq b$ for all $b \in (\pi/2, \pi)$, that is, $\gamma(s) \leq \pi/2$. If (a_0, a_1) is the largest subinterval of $[0, \infty]$ (of r) such that the equation $\gamma(s) = \pi/(2r)$ has a (necessarily unique) solution for s, say $s = s_0(r) > 0$, then $\pi/(2r) = \gamma(s_0(r)) \leq \pi/2$, that is, $r \geq 1$, and therefore $a_1 > 1$. *We shall assume that* $a_1 > 1$. For simplicity, we shall also *assume* that $V(\zeta)D(S) \subset D(S)$ for all $\zeta \in \mathbb{C}^+$, and that the X-valued function $SV(\cdot)x$ is *strongly continuous* in $\mathbb{C}^+$ for each $x \in D(S)$.

Theorem 3.11. *Let iS be the generator of the C_o-group $S(\cdot)$. Let $V(\cdot)$ be a regular semigroup such that*

(a) $a_1 > 1$*;*
(b) $V(\zeta)D(S) \subset D(S)$ *for* $\zeta \in \mathbb{C}^+$*, and* $SV(\cdot)x$ *is (strongly) continuous in* $\mathbb{C}^+$ *for each* $x \in D(S)$*; and*
(c) $V := V(1)$ *satisfies the Volterra relation with* S*.*

Then

(i) $[S, V(\zeta)]x = \zeta V(\zeta+1)x$ *for all* $x \in D(S)$ *and* $\zeta \in \mathbb{C}^+$*; and*
(ii) $S + zV$ *is similar to* $S + wV$ $(z, w \in \mathbb{C})$ *if* $\Re z = \Re w$*; specifically,*

$$S + zV = V(i\eta)^{-1}(S + wV)V(i\eta), \tag{17}$$

where $i\eta = z - w = i\Im(z - w)$.

(Conversely, independently of Condition (a), (i) implies Conditions (b) and (c).)

Proof. We first verify the simple (parenthetic) closing observation in the statement of the theorem. If $x \in D(S)$, then for all $\zeta \in \mathbb{C}^+$, the identity (i) includes implicitly the requirement $V(\zeta)x \in D(S)$ (so that $[S, V(\zeta)]x$ makes sense!), hence indeed $V(\zeta)\,D(S) \subset D(S)$, and moreover, by (i), the function

$$\zeta \to S\,V(\zeta)x = V(\zeta)(Sx) + \zeta V(\zeta+1)x$$

is clearly (strongly) continuous in $\mathbb{C}^+$.

Condition (c) is the special case of (i) with $\zeta = 1$.

We prove now the main statement of the theorem.

Since $a_1 > 1$, we may take $\alpha = 1$ in Theorem 17.6.1 in [HP]. We then have the series representation

$$V(\zeta) = \sum_{n=0}^{\infty} \binom{\zeta}{n} (V - I)^n \quad (\zeta \in \mathbb{C}^+),$$

with strong convergence at least for $\Re\zeta > s_0(1)$, where $s_0(1) := \gamma^{-1}(\pi/2) > 0$. (Cf. (16).)

Denote

$$V_k(\zeta) := \sum_{n=0}^{k} \binom{\zeta}{n} (V - I)^n \quad (\zeta \in \mathbb{C},\ k = 0, 1, \dots). \tag{18}$$

By induction, it follows from the Volterra relation for V with S that

$$[S, V^j]x = jV^{j-1}V^2x \quad (j \in \mathbb{N},\ x \in D(S)), \tag{19}$$

and therefore, by linearity,

$$p(V)D(S) \subset D(S);\ [S, p(V)]x = p'(V)V^2x \quad (x \in D(S)), \tag{20}$$

for any polynomial p. In particular, for the polynomial $p(z) = (z-1)^n$,

$$(V - I)^n D(S) \subset D(S);\ [S, (V - I)^n]x = n(V - I)^{n-1}V^2x$$

for all $x \in D(S)$ and $n \in \mathbb{N}$. Hence, for each $x \in D(S)$, $k \in \mathbb{N}$, and $\Re\zeta - 1 > s_o(1)$, we have $V_k(\zeta)x \in D(S)$, $V_k(\zeta)x \to V(\zeta)x$, and (using the identity $\binom{\zeta}{n}n = \binom{\zeta-1}{n-1}\zeta$)

$$\begin{aligned} SV_k(\zeta)x &= V_k(\zeta)Sx + \sum_{n=1}^{k} \binom{\zeta}{n} n(V-I)^{n-1}V^2x \\ &= V_k(\zeta)Sx + \zeta V_{k-1}(\zeta-1)V^2x \\ &\quad \to V(\zeta)Sx + \zeta V(\zeta-1)V^2x = V(\zeta)Sx + \zeta V(\zeta+1)x \end{aligned}$$

as $k \to \infty$. Since S is closed, it follows that

$$V(\zeta)D(S) \subset D(S)$$

and

$$SV(\zeta)x = V(\zeta)Sx + \zeta V(\zeta+1)x \quad (x \in D(S);\ \Re\zeta > s_o(1)+1). \tag{21}$$

Let Γ be any Jordan path in $\mathbb{C}^+$. If $J_n x$, $n \in \mathbb{N}$, $x \in D(S)$ are Riemann sums for the integral $\int_\Gamma V(\zeta)x\,d\zeta$, then $J_n x \in D(S)$ and $J_n x \to \int_\Gamma V(\zeta)x\,d\zeta$ (strongly), because $V(\zeta)D(S) \subset D(S)$ by hypothesis and $V(\cdot)x$ is continuous in $\mathbb{C}^+$. Also $SJ_n x$ are Riemann sums for $\int_\Gamma SV(\zeta)x\,d\zeta$, by the linearity of S, and consequently their (strong) limit as $n \to \infty$ is equal to the latter integral, by the assumed continuity of $SV(\cdot)x$ in $\mathbb{C}^+$. Since S is closed, it follows that

$$\int_\Gamma V(\zeta)x\,d\zeta \in D(S)$$

and

$$S\int_\Gamma V(\zeta)x\,d\zeta = \int_\Gamma SV(\zeta)x\,d\zeta. \tag{22}$$

In particular, for any (closed) triangular path $\Gamma \subset \mathbb{C}^+$, the analyticity of $V(\cdot)$ in $\mathbb{C}^+$ implies that the left-hand side of (22) vanishes, and therefore, by (the vector version of) Morera's theorem, $SV(\cdot)x$ is analytic in $\mathbb{C}^+$ for each $x \in D(S)$. Since both sides of (21) are analytic in $\mathbb{C}^+$ and coincide for $\Re\zeta > s_o(1)+1$, we conclude that (21) is valid *for all* $\zeta \in \mathbb{C}^+$ and $x \in D(S)$, and (i) is proved.

Fix $t \in \mathbb{R}$ and $x \in D(S)$. For any $s > 0$, $V(s+it)x \in D(S)$, $V(s+it)x \to V(it)x$ as $s \to 0$, and by (21) and our hypothesis,

$$SV(s+it)x = V(s+it)Sx + (s+it)V(s+1+it)x \to V(it)Sx + itV(1+it)x.$$

Since S is closed, it follows that $V(it)x \in D(S)$ and

$$SV(it)x = V(it)Sx + itV(1+it)x = V(it)(S+itV)x. \tag{23}$$

Thus, on $D(S)$,

$$S + itV = V(-it)S\,V(it),$$

and therefore, *for any* $s \in \mathbb{R}$,

$$S + (s+it)V = (S + itV) + sV = V(-it)S\,V(it) + sV$$
$$= V(-it)(S + sV)V(it) \qquad (24)$$

on $D(S)$. For any $u \in \mathbb{R}$, $V(-iu)x \in D(S)$ if $x \in D(S)$, and it follows from (24) that

$$V(iu)(S + (s+iu)V)V(-iu)x = (S + sV)x,$$

and consequently, again by (24),

$$T_{s+it}x = V(-it)T_s V(it)x = V(-it)V(iu)T_{s+iu}V(-iu)V(it)x,$$

for all $x \in D(S) = D(T_{s+it})$, that is,

$$T_{s+it} \subset V(-i(t-u))T_{s+iu}V(i(t-u)) \quad (s,t,u \in \mathbb{R}). \qquad (25)$$

Interchanging t and u, we get

$$T_{s+iu} \subset V(-i(u-t))T_{s+it}V(i(u-t)),$$

hence

$$V(-i(t-u))T_{s+iu}V(i(t-u)) \subset T_{s+it}$$

(because $V(i(t-u))D(S) \subset D(S)$). We then have equality in (25), and (ii) is proved. □

We shall need an upper and a lower estimate on the norms $\|T_{s+it}(u)\|$ $(s,t,u \in \mathbb{R})$ in order to prove that the sufficient condition for similarity in Theorem 3.11 is also ("almost") necessary. These estimates are the content of the following two lemmas.

Lemma 3.12 (Hypothesis as in Theorem 3.11.). *There exists a constant* $H \geq 1$ *such that*

$$\|T_{s+it}(u)\| \leq H\,\|S(u)\|\,(1 + |u|\,\|V\|)^s e^{2b|t|}$$

for all $s,t,u \in \mathbb{R}$.

Proof. Observe first that if A and B generate the C_o-semigroups $S(\cdot)$ and $T(\cdot)$, respectively, and $Q \in B(X)$ is nonsingular, then $A = Q^{-1}BQ$ if and only if $S(\cdot) = Q^{-1}T(\cdot)Q$ (the verification of this statement is an easy exercise). It then follows from Theorem 3.11 that

$$T_{s+it}(u) = V(-it)T_s(u)V(it) \quad (s,t,u \in \mathbb{R}). \qquad (26)$$

Fix $u \in \mathbb{R}$, and define

$$F_u(z) := e^{bz^2}T_z(u) \quad (z \in \mathbb{C}). \qquad (27)$$

Then by (26)

$$\|F_u(s+it)\| \leq C^2\|T_s(u)\|\, e^{b(s^2-t^2+2|t|)}$$
$$\leq C^2\|T_s(u)\|\, e^{b(s^2+1)} \quad (s,t,u \in \mathbb{R}). \tag{28}$$

Therefore $F_u(s+it)$ is bounded in each vertical strip $k-1 \leq s \leq k$ $(k \in \mathbb{Z})$. On the lines $s=k$, we have by (28) and both identities of Lemma 3.8

$$F_u(k+it) \leq C^2\|S(u)\|(1+|u|\,\|V\|)^{|k|}e^{b(k^2+1)} \quad (t \in \mathbb{R},\, k \in \mathbb{Z}). \tag{29}$$

By Theorem 1.67 (with $A=iS$ or $A=-iS$, and with $B(z)=zV$), the $B(X)$-valued function $F_u(\cdot)$ is entire. We may then apply to it the (operator version of the) "three lines theorem" (cf. [DS I–III, Theorem VI.10.3]) in each strip $k-1 \leq s \leq k$ $(k \in \mathbb{Z})$. Writing

$$s = p(k-1)+(1-p)k = k-p$$

with $0 \leq p \leq 1$, so that also $|s| = p|k-1|+(1-p)|k|$, we get from (29)

$$\|F_u(s+it)\| \leq C^2e^b\|S(u)\|(1+|u|\,\|V\|)^{|s|}e^{b(p(k-1)^2+(1-p)k^2)} \tag{30}$$

for all $s,t,u \in \mathbb{R}$. However,

$$p(k-1)^2+(1-p)k^2 = (k-p)^2+p(1-p) = s^2+p(1-p) \leq s^2+1/4,$$

and therefore, by (30) with $t=0$

$$\|T_s(u)\| = e^{-bs^2}\|F_u(s)\| \leq C^2e^{5b/4}\|S(u)\|(1+|u|\,\|V\|)^{|s|},$$

and the lemma follows now from (26) and the estimate $\|V(it)\| \leq Ce^{b|t|}$, with $H = C^4e^{5b/4}$. □

For the lower estimate, we shall consider only the case of a uniformly bounded C_o-group $S(\cdot)$.

Lemma 3.13 (Hypothesis as in Theorem 3.11). *Suppose in addition that $\|S(u)\| \leq M$ for all $u \in \mathbb{R}$. Then there exists a strictly positive function $C(\cdot)$ on $\mathbb{R}$ such that*

$$\|T_{s+it}(u)\| \geq C(s)(1+|u|\,\|V\|)^{|s|}e^{-2b|t|}$$

for all $s,t,u \in \mathbb{R}$.

Proof. Observe first that $V \neq 0$ (otherwise $V(z)=0$ for $|\Re z|>1$, and therefore $V(\cdot)=0$ in $\mathbb{C}^+$ by analyticity. But then $V(it)=0$ for $t \in \mathbb{R}$, which is absurd because $V(it)$ are nonsingular).

By (26), for all $s,t,u\in\mathbb{R}$,

$$\|T_s(u)\| = \|V(it)T_{s+it}(u)V(-it)\| \\ \le C^2 e^{2b|t|}\|T_{s+it}(u)\|.$$

Hence

$$\|T_{s+it}(u)\| \ge C^{-2}e^{-2b|t|}\|T_s(u)\| \quad (s,t,u\in\mathbb{R}). \tag{31}$$

Denote

$$B_u(s) := (1+|u|\,\|V\|)^{-s}\|T_s(u)\|$$

and

$$B(s) := \inf_{u\in\mathbb{R}} B_u(s) \quad (s\in\mathbb{R}).$$

The estimate of the lemma (with $C(s) := C^{-2}B(s)$) follows from (31), and *all that remains to prove is that $B(s)\neq 0$ for all $s\in\mathbb{R}$.*

The hypothesis $\|S(u)\|\le M$ for all $u\in\mathbb{R}$ implies that the types of both semigroups $\{S(u); u\ge 0\}$ and $\{S(-u); u\ge 0\}$ are zero. It then follows from Theorem 1.4 that

$$B_u(0) = \|S(u)\| \ge r(S(u)) = 1 \quad (u\in\mathbb{R}),$$

and since $B_0(0) = \|S(0)\| = 1$, we conclude that $B(0)=1$.

By Corollary 3.9, we have the identities

$$T_{s+1}(u) = T_s(u)(I+iuV);\ T_{s-1}(u) = (I-iuV)T_s(u) \quad (s,u\in\mathbb{R}).$$

Therefore

$$B_u(s+1)\le B_u(s) \quad (s\ge 0)$$

and

$$B_u(s-1)\le B_u(s) \quad (s\le 0).$$

Hence

$$B(s+1)\le B(s) \quad (s\ge 0)$$

and

$$B((s-1)\le B(s) \quad (s\le 0).$$

It then suffices to prove that $B(s)\neq 0$ for $|s|>1$.

Proceeding by contradiction, suppose $B(s_0)=0$ for some real s_0 with $|s_0|>1$. Let then $\{u_k\}\subset\mathbb{R}$ be such that $B_{u_k}(s_0)\to 0$. If the sequence $\{u_k\}$ is bounded, we may pick a subsequence $\{u'_k\}$ converging to some $u_0\in\mathbb{R}$; then for all $x\in X$,

$$\|T_{s_0}(u_0)x\| = \lim_k \|T_{s_0}(u'_k)x\| = \lim_k (1+|u'_k|)^{s_0} B_{u'_k}(s_0)x = 0,$$

that is, $T_{s_0}(u_0)=0$, which is absurd since $T_{s_0}(u_0)$ is nonsingular (as a "value" of the group $T_{s_0}(\cdot)$). Thus $\{u_k\}$ is unbounded, and *we may assume that* $|u_k|\to\infty$.

Let $\epsilon > 0$, and let k_0 be chosen such that

$$B_{u_k}(s_0) < \epsilon^{|s_0|} \tag{32}$$

for all $k > k_0$. Fix such a k, and consider the entire functions

$$G_k^{\pm}(z) := (1 + |u_k|\,\|V\|)^{\mp z} F_{u_k}(z)$$

(cf. (27)). They are bounded in each closed vertical strip. By (28) and (32), for all $t \in \mathbb{R}$,

$$\|G_k^{\pm}(s_0 + it)\| \le C^2 e^{b(s_0^2+1)} (1 + |u_k|\,\|V\|)^{\mp s_0 + |s_0|} \epsilon^{|s_0|}.$$

Taking G_k^+ if $s_0 > 0$ and G_k^- if $s_0 < 0$, we get

$$\|G_k^{\pm}(s_0 + it)\| \le C^2 e^{b(s_0^2+1)} \epsilon^{|s_0|}. \tag{33}$$

Since $\|S(\cdot)\| \le M$ by hypothesis, it follows from (28) that

$$\|G_k^{\pm}(it)\| \le C^2 M\, e^b \quad (t \in \mathbb{R}). \tag{34}$$

We now apply the three lines theorem to G_k^+ in the strip $0 \le \Re z \le s_0$ if $s_0 > 0$, or to G_k^- in the strip $s_0 \le \Re z \le 0$ if $s_0 < 0$. By (33) and (34) (since $M \ge 1$), we obtain

$$\|G_k^{\pm}(s + it)\| \le C^2 M\, e^{b(s_0^2+1)} \epsilon^s$$

for all $s + it$ in the respective strips.

Since $|s_0| > 1$, these estimates are valid for $s + it = 1$ (or -1, respectively), that is,

$$(1 + |u_k|\,\|V\|)^{-1} \|T_{\pm 1}(u_k)\| \le C^2 M\, e^{b(s_0^2+1)} \epsilon$$

for all $k > k_0$. Hence

$$\lim_k (1 + |u_k|\,\|V\|)^{-1} \|T_{\pm 1}(u_k)\| = 0.$$

Since $\|S(\cdot)\| \le M$, it now follows from Lemma 3.8 that

$$\lim_k (1 + |u_k|\,\|V\|)^{-1} \|I \pm i u_k V\| = 0.$$

On the other hand, since $V \neq 0$ (as observed above) and $|u_k| \to \infty$, the limit is clearly equal to 1, contradiction. □

Corollary 3.14 (Hypothesis as in Lemma 3.13). *If $z, w \in \mathbb{C}$ are such that $|\Re z| \neq |\Re w|$, then T_z is not similar to T_w.*

Proof. Write $s = \Re z$ and $r = \Re w$, and assume without loss of generality that $|s| > |r|$. By Theorem 3.11, T_z and T_w are similar to T_s and T_r, respectively, and it suffices then to prove that T_s is *not similar* to T_r. Proceeding by contradiction, suppose there exists a nonsingular $Q \in B(X)$ such that $T_s = Q^{-1}T_rQ$. Then $T_s(u) = Q^{-1}T_r(u)Q$ for all $u \in \mathbb{R}$. By Lemmas 3.12 and 3.13, we have for all $u \in \mathbb{R}$

$$\begin{aligned} 0 < C(s) &\le (1 + |u|\,\|V\|)^{-|s|}\|T_s(u)\| \\ &\le (1 + |u|\,\|V\|)^{-|s|}\|Q\|\,\|Q^{-1}\|\,\|T_r(u)\| \\ &\le H\,M\|Q\|\,\|Q^{-1}\|\,(1 + |u|\,\|V\|)^{|r|-|s|} \to 0 \end{aligned}$$

as $|u| \to \infty$ (since $\|V\| > 0$ and $|r| - |s| < 0$), contradiction. □

This last corollary and Theorem 3.11 yield the following main result of the section:

Theorem 3.15 (Hypothesis as in Theorem 3.11 and $\|S(\cdot)\| = O(1)$). *Let $z, w \in \mathbb{C}$. Then T_z is similar to T_w if $\Re z = \Re w$ and only if $|\Re z| = |\Re w|$.*

We state in particular the interesting special case with $w = 0$:

Corollary 3.16 (Hypothesis as in Theorem 3.15). *T_z is similar to S if and only if $\Re z = 0$.*

Example 3.17.

1. Let $V(\cdot)$ be the *Gamma semigroup* on $L^p(\mathbb{R}^+)$, $1 < p < \infty$, as defined in Example 1.116. We observed there that its boundary group $V(it)$ satisfies the growth condition

$$\|V(it)\| \le C(1 + |t|)e^{\pi|t|/2} \quad (t \in \mathbb{R}) \tag{35}$$

(and the factor $C(1 + |t|)$ can be omitted in case $p = 2$). Hence, for any $r \in (\pi/2, \pi)$, $\|V(it)\| \le C_r e^{r|t|}$ $(t \in \mathbb{R})$ for a suitable constant C_r, and we observed before the statement of Theorem 3.11 that the latter estimate implies Condition (a) of Theorem 3.11.
If S denotes the multiplication operator

$$(Sf)(s) = sf(s) \quad (s \in \mathbb{R}^+)$$

with maximal domain in $L^p(\mathbb{R}^+)$, then iS is clearly the generator of the C_o-group of isometries $S(t) : f(s) \to e^{its}f(s)$ $(t \in \mathbb{R};\ s \in \mathbb{R}^+)$, and a simple direct calculation verifies the identity (i) in Theorem 3.11 (hence, by the parenthetical statement there, both Conditions (b) and (c) of the theorem are satisfied). It then follows from Theorem 3.15 that $S + z\,V$ is similar to $S + w\,V$ if $\Re z = \Re w$ and only if $|\Re z| = |\Re w|$. (*In particular, $S + z\,V$ is similar to S if and only if $\Re z = 0$.*)

2. On the space $L^p(0,N)$ $(1 < p < \infty;\ N < \infty)$, we may take $V(\cdot)$ to be the Gamma semigroup with parameter $b = 0$, that is, the Riemann–Liouville semigroup (cf. Example 1.116). Since $V(it)$ satisfies the growth property (35), the conditions of Theorem 3.15 are satisfied as before, and so is therefore its conclusion. Moreover, in the present example, the conclusion can be strengthened to the statement (cf. [K4, Corollary 9.3]):
$S + zV$ *is similar to* $S + wV$ *if and only if* $\Re z = \Re w$.

B.3 Similarity of Certain Perturbations

In the setting of Corollary 3.16, $S - V$ is *not similar* to S. We look for perturbations $(S - V) + P$ that *are similar* to S (for example, this is trivially true for $P = V$). We shall prove presently that, *without any restriction on* V *and on the* C_o*-group* $S(\cdot)$ *generated by* iS, the Volterra relation alone (between S and V) implies that the perturbations $(S - V) + P$ of S are similar to S *for all* P *in the* V*-similarity suborbit* given by

$$\mathcal{V} := \{S(a)V\,S(-a);\ a \in \mathbb{R}\}. \tag{1}$$

In particular, if S is a (not necessarily bounded) scalar-type spectral operator with real spectrum (as in Example 3.17), and $V \in B(X)$ satisfies the Volterra relation with S, then $(S - V) + P$ is a scalar-type spectral operator with real spectrum (similar to S) for all P of the form $e^{iaS}V\,e^{-iaS}$ with $a \in \mathbb{R}$ (the exponential is defined by means of the resolution of the identity for S).

When S is a *bounded* operator, the above perturbations are similar to S for all P in the wider V-similarity suborbit $\mathcal{W}$ given by

$$\mathcal{W} := \{e^{\lambda S}V\,e^{-\lambda S};\ \lambda \in \mathbb{C}\}. \tag{2}$$

This result is actually true in a Banach algebra context.

(Note that in case S is a *bounded spectral operator*, the above perturbations are then trivially spectral of the same type as S and have spectrum equal to $\sigma(S)$, without any restriction on the "type" or the spectrum of the spectral operator S.)

Theorem 3.18. *Let* iS *be the generator of the* C_o*-group of operators* $S(\cdot)$ *on the Banach space* X*, and let* $V \in B(X)$ *satisfy the Volterra relation with* S *(that is,* $V\,D(S) \subset D(S)$ *and* $[S, V] \subset V^2$*). Then the perturbations* $(S-V)+P$ *of* $S - V$ *are similar to* S *for all* P *in the* V*-similarity suborbit*

$$\mathcal{V} := \{S(a)V\,S(-a);\, a \in \mathbb{R}\}.$$

Proof. Fix $V_a := S(a)V\,S(-a) \in \mathcal{V}$. Note that $D(S)$ is V_a-invariant, since it is invariant for V and for $S(t)$ for all $t \in \mathbb{R}$ (cf. Theorem 1.2).

By Lemma 3.7 (e),

$$iaVV_a = (iaV\,S(a)V)\,S(-a) = [S(a), V]\,S(-a) = V_a - V \tag{3}$$

and

$$iaV_aV = -S(a)(i(-a)V\,S(-a)V) = -S(a)\,[S(-a),V] = V_a - V. \tag{4}$$

Thus V_a *commutes* with V, and by (3)

$$\begin{aligned}(I - iaV)(I + iaV_a) &= I - iaV + iaV_a - ia\,(iaVV_a)\\ &= I + ia(V_a - V) - ia(V_a - V) = I.\end{aligned}$$

Therefore $I - iaV$ is *nonsingular*, and

$$(I - iaV)^{-1} = I + iaV_a. \tag{5}$$

Consider now the perturbation of $S - V$ by V_a, that is, the operator

$$T_a := (S - V) + V_a \tag{6}$$

with domain $D(S)$. By the Volterra relation and (4), we have

$$T_a(I - iaV) = T_a - ia(SV - V^2) - (V_a - V) = S - iaVS = (I - iaV)S.$$

Since $(I - iaV)$ is nonsingular, it follows that

$$(I - iaV)^{-1}T_a(I - iaV) = S \tag{7}$$

on $D(S)$. However, if x is in the domain of the operator on the left-hand side of (7), then necessarily $(I - iaV)x \in D(T_a) = D(S)$, and therefore, by (5) and the V_a-invariance of $D(S)$ (cf. observation at the beginning of the proof),

$$x = (I + iaV_a)\,[(I - iaV)x] \in D(S).$$

This shows that (7) is valid *with equality of domains.* □

Corollary 3.19. *Let $S, V \in B(X)$ satisfy the Volterra relation $[S, V] = V^2$. Then the perturbations $(S - V) + P$ are similar to S for all P in the V-similarity suborbit*

$$\mathcal{W} := \{e^{\lambda S}V\,e^{-\lambda S};\ \lambda \in \mathbb{C}\}.$$

Proof. By Theorem 3.18,

$$[(S - V) + e^{iaS}V\,e^{-iaS}]\,(I - iaV) = (I - iaV)S \tag{8}$$

for all $a \in \mathbb{R}$. Replacing ia by any complex number λ, both sides of the equation are entire $B(X)$-valued functions of λ, and they coincide on $i\mathbb{R}$ (by (8)). Therefore (8) is valid with ia replaced by arbitrary $\lambda \in \mathbb{C}$. □

Miscellaneous Exercises

Abstract Landau Inequality

1. Let A be the generator of the C_o-semigroup $T(\cdot)$.

(a) Verify the identity

$$t^{-1}(T(t)x - x) = Ax + t^{-1}\int_0^t (t-s)T(s)A^2x\,ds \quad (x \in D(A^2). \tag{1}$$

(b) If $\|T(\cdot)\| \le M$, show that

$$\|Ax\| \le M(2\|x\|/t + (t/2)\|A^2x\|) \quad (t>0;\ x \in D(A^2)). \tag{2}$$

(c) If $\|T(\cdot)\| \le M$, prove that

$$\|Ax\| \le 2M(\|A^2x\|\,\|x\|)^{1/2} \quad (x \in D(A^2)). \tag{3}$$

(Hint: minimize the right-hand side of (2) with respect to t.)

(d) Formulate (c) for the translation semigroup on $L^p(\mathbb{R})$, $1<p<\infty$. (*This special case is known as Landau's inequality.*)

Variation on the Theme of Dissipativity

Notation. Given an operator A with domain $D(A)$, we shall denote by $D(A)_1$ the unit sphere of $D(A)$:

$$D(A)_1 := \{x \in D(A);\ \|x\|=1\}.$$

2. Suppose the operator A is *weakly dissipative* (w.d.), that is,
For each $x \in D(A)_1$, there exists a unit vector $x^* \in X^*$ such that $x^*x = 1$ and $\Re x^*Ax \le 0$.

(a) If A is w.d., show that it is *bounded below* (b.b.), that is,

$$\|(\lambda I - A)x\| \geq \lambda\|x\| \quad (x \in D(A);\ \lambda > 0). \tag{4}$$

(b) Denote $x_\lambda := (\lambda I - A)x$ for each $x \in D(A)$ and $\lambda > 0$. For each $x \in D(A)_1$ and $\lambda > 0$, there exists a unit vector $x_\lambda^* \in X^*$ such that

$$x_\lambda^* x_\lambda = \|x_\lambda\|. \tag{5}$$

If A is b.b., prove that such a vector x_λ^* satisfies necessarily the following inequalities:

$$\Re x_\lambda^* Ax \leq 0; \quad \Re x_\lambda^* x \geq 1 - \|Ax\|/\lambda. \tag{6}$$

Since the norm-closed unit ball of X^* is *weak**-compact, $\{x_\lambda^*\}$ has a *weak** limit point x^* as $\lambda \to \infty$. Verify that $\|x^*\| = 1$ and $\Re x^* Ax \leq 0$, that is, *A is w.d. iff it is b.b.*

(c) If A is b.b., then for each $\lambda > 0$, $\lambda I - A$ is injective and

$$\|\lambda(\lambda I - A)^{-1}x\| \leq \|x\| \quad (x \in D((\lambda I - A)^{-1})).$$

(d) If A is b.b. and $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$, prove that $\mathbb{R}^+ \subset \rho(A)$.

(e) Let A be b.b.; prove: (i) A is a closed operator iff $\lambda I - A$ has a closed range for some (hence for all) $\lambda > 0$. (ii) If A is densely defined, then it is *closable*, $\overline{A}$ is b.b., the range of $\lambda I - \overline{A}$ is the closure of the range of $\lambda I - A$ (for all $\lambda > 0$), and the latter range is dense in X for some $\lambda > 0$ iff $\overline{A}$ generates a contraction C_o-semigroup.

Resolvents of the Hille–Yosida Approximations

3. Let A be the generator of the C_o-semigroup $T(\cdot)$ with type ω, and let A_λ be its Hille–Yosida approximation ($\lambda > \omega$). Prove that

$$R(\mu; A) = \lim_{\lambda \to \infty} R(\mu; A_\lambda) \tag{7}$$

in the s.o.t., uniformly in μ on any vertical segment $\{\mu = c + it;\ -\tau \leq t \leq \tau\}$ with $c > \omega$ fixed. (Hint: $T_\lambda(\cdot) \to T(\cdot)$, cf. proof of Theorem 1.17.)

Adjoint Semigroup

4. Let A be the generator of the C_o-semigroup $T(\cdot)$ on the Banach space X. Since $D(A)$ is dense, the adjoint A^* is well-defined. Let W be *the closure of $D(A^*)$ in X^**. Prove:

(a) W is $T(t)^*$-invariant for all $t \geq 0$, and $R(\lambda; A^*)$-invariant for all $\lambda > \omega$ (where ω is the type of $T(\cdot)$).
Denote

$$S(t) := T(t)^*|_W; \quad R(\lambda) := R(\lambda; A^*)|_W \quad (t \geq 0; \lambda > \omega). \tag{8}$$

(b) Prove that $R(\cdot) : (\omega, \infty) \to W$ is a pseudo-resolvent with range dense in W, and

$$\|[(\lambda - \omega)R(\lambda)]^n\| \leq M \quad (\lambda > \omega;\ n \in \mathbb{N}). \tag{9}$$

Consequently (cf. Theorems 1.14 and 1.17), $R(\cdot)$ is the resolvent of the generator B of a C_o-semigroup $U(\cdot)$ on W. Prove that $U(\cdot) = S(\cdot)$. (Cf. Theorem 1.36.)
(c) Denote by A^*_W *the part of* A^* *in* W (cf. Definition 1.19). Prove that $B = A^*_W$.
(d) If X is reflexive, show that $W = X^*$, and $T(\cdot)^*$ is a C_o-semigroup whose generator is A^*.

Spectra of a Semigroup and its Generator

5. Let A be the generator of the C_o-semigroup $T(\cdot)$. Consider the *truncated Laplace transform* of $T(\cdot)$, defined by

$$L_r(\lambda)x = \int_0^r e^{-\lambda t} T(t)x\, dt \quad (r > 0; x \in X; \lambda \in \mathbb{C}). \tag{10}$$

Clearly $L_r(\lambda) \in B(X)$.

(a) Prove the identity

$$L_r(\lambda)(\lambda I - A) \subset (\lambda I - A)L_r(\lambda) = I - e^{-\lambda r}T(r). \tag{11}$$

(b) If $e^{\lambda r} \in \rho(T(r))$, then $\lambda \in \rho(A)$, and

$$R(\lambda; A) = e^{\lambda r} L_r(\lambda) R(e^{\lambda r}; T(r)). \tag{12}$$

In particular,

$$e^{r\sigma(A)} \subset \sigma(T(r)) \quad (r \geq 0). \tag{13}$$

(c) Prove that

$$e^{r\sigma_p(A)} \subset \sigma_p(T(r)) \quad (r \geq 0). \tag{14}$$

(Hint: apply Part (a).)
(d) Suppose $T(\cdot)$ is differentiable in the s.o.t. in (b, ∞) for some $b \geq 0$. Prove that $AT(t) \in B(X)$ and commutes with $T(s)$ for all $t > b$ and $s \geq 0$.

(e) Let $T(\cdot)$ be as in Part (d), and fix $r > b$. Prove that if $\lambda e^{r\lambda} \in \rho(AT(r))$, then $\lambda \in \rho(A)$ and

$$R(\lambda; A) = [T(r) + \lambda e^{r\lambda} L_r(\lambda)]\, R(\lambda e^{r\lambda}; AT(r)), \tag{15}$$

and the factors in the latter product may be interchanged.

(f) ($T(\cdot)$ as in Part (d).) Fix $r > b$, $\delta > 0$, and a, M such that $\|T(t)\| \le M\, e^{at}$ for all $t \ge 0$ (cf. Theorem 1.1). Denote $c = (1+\delta)\|AT(r)\|$ and

$$\Omega_{c,r} = \{\lambda \in \mathbb{C};\, c\, e^{-r\Re\lambda} \le |\Im\lambda|\}. \tag{16}$$

Prove that $\Omega_{c,r} \subset \rho(A)$ and

$$\sup_{\lambda \in \Omega_{c,r};\, \Re\lambda \le a} \|(\Im\lambda)^{-1} R(\lambda; A)\| < \infty. \tag{17}$$

(*Conversely*, if there exist positive constants c, r with the latter property, then $T(\cdot)$ is differentiable in the s.o.t. in some ray (b, ∞). See [P].)

Compact Semigroups

6. Let $T(\cdot)$ be a C_o-semigroup such that $T(t)$ is a compact operator for each $t > t_0$. Fix $t > t_0$ and $\epsilon > 0$. Denote $M := \sup_{0 \le s \le 1} \|T(s)\|$. By compactness of the set $\{T(t)x; \|x\| \le 1\}$, choose a finite open cover of it by open balls $B(T(t)x_j, \epsilon/(2(M+1)))$, $j = 1, \ldots, n$. By continuity of $T(\cdot)x_j$ at t, choose $0 < \delta < 1$ such that

$$\|T(t+h)x_j - T(t)x_j\| < \epsilon/2 \quad (0 < h < \delta;\; j = 0, \ldots, n). \tag{18}$$

(a) Prove that

$$\|T(t+h) - T(t)\| < \epsilon \quad (0 < h < \delta) \tag{19}$$

and conclude that $T(\cdot)$ is continuous in the u.o.t. for $t > t_0$ (for $h < 0$, use the argument in the proof of Theorem 1.1).

(b) If $t_0 = 0$, $R(\lambda; A)$ is compact for all $\lambda \in \rho(A)$. (Hint: for $\Re\lambda > \omega$, apply Theorem 1.15; for arbitrary $\lambda \in \rho(A)$, use then the resolvent identity.)

(c) If $T(\cdot)$ is continuous in the u.o.t. on $(0, \infty)$, show that

$$T(t) = \lim_{\lambda \to \infty} \lambda R(\lambda; A)\, T(t) \quad (t > 0) \tag{20}$$

in the $B(X)$-norm.

(d) Conclude: $T(\cdot)$ *is compact on* $(0, \infty)$ *if and only if it is continuous in the u.o.t. on* $(0, \infty)$ *and* $R(\lambda; A)$ *is compact for all* $\lambda \in \rho(A)$.

(e) If $t_0 = 0$ and $B \in B(X)$, then $A + B$ generates a C_o-semigroup $S(\cdot)$ which is compact on $(0, \infty)$. (Cf. Lemmas 2 and 5 in the proof of Theorem 1.38.)

Powers of the Generator

7. Let $-A$ be the generator of a C_o-semigroup $T(\cdot)$ which satisfies the growth condition

$$\|T(t)\| \leq M\, e^{-at} \quad (t \geq 0) \tag{21}$$

for some $a > 0$ (note the minus sign!). Let $\alpha \geq 0$, and define the operator $A^{-\alpha}$ by $A^0 := I$ and

$$A^{-\alpha} x := \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} T(t) x\, dt \quad (x \in X;\ \alpha > 0). \tag{22}$$

(a) Prove that the above integral converges uniformly on the closed unit ball of $B(X)$ (therefore $A^{-\alpha} \in B(X)$).

(b) Prove that

$$A^{-(\alpha+\beta)} = A^{-\alpha} A^{-\beta} \quad (\alpha, \beta \geq 0). \tag{23}$$

(Hint: use the identity

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\, du.) \tag{24}$$

In particular, for all $n \in \mathbb{N}$, $A^{-n} = (A^{-1})^n$, the usual n-th power of $A^{-1} \in B(X)$ (cf. Theorem 1.15).

(c) For $0 < \alpha < 1$, prove the identity

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha} R(\lambda; -A)\, d\lambda, \tag{25}$$

where the integral converges in $B(X)$. Hint: use the identity

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}.$$

In particular,

$$\|A^{-\alpha}\| \leq M(1 + a^{-1}) \quad (\alpha \in (0,1)). \tag{26}$$

(d) Prove that the function $t \in [0, \infty) \to A^{-t}$ is a C_o-semigroup.

(e) Prove that $A^{-\alpha}$ is injective for all $\alpha \geq 0$.
Using (e), we *define*

$$A^\alpha := (A^{-\alpha})^{-1}. \tag{27}$$

Prove that the (closed) operators A^α (with domain equal to the range of $A^{-\alpha}$) have the following properties:

(1) If $0 < \alpha \leq \beta$, then $D(A^\alpha) \subset D(A^\beta)$.

(2) For each $\alpha \geq 0$, $D(A^\alpha)$ is dense in X.

(3) If $\alpha, \beta \in \mathbb{R}$, then

$$A^{\alpha+\beta} x = A^\alpha A^\beta x \tag{28}$$

for all $x \in D(A^\gamma)$, where $\gamma = \max(\alpha, \beta, \alpha+\beta)$.

C^∞-semigroups

8. Let A be the generator of a C_o-semigroup $T(\cdot)$ such that $T(t)X \subset D(A)$ for all $t > 0$.

(a) Prove the identity

$$T(t+h)x - T(t)x = \int_0^h T(s)AT(t)x\,ds \quad (x \in X;\, t, h > 0). \tag{29}$$

(b) Prove that $T(\cdot)$ is C^∞ on $(0, \infty)$ in the u.o.t. Hint: use (a) and the fact that for all $t > 0$, $AT(t) \in B(X)$ (why?). Conclude that $T'(t)$ exists (in the u.o.t.) and is equal to $AT(t) = T(t-a)AT(a)$ for all $t > a > 0$, etc.

Entire Vectors

9. Let $T(\cdot)$ be a C_o-group. Given $x \in X$, define

$$x_n := (n/2\pi)^{1/2} \int_{\mathbb{R}} e^{-nt^2/2} T(t)x\,dt \quad (n \in \mathbb{N}); \tag{30}$$

$$x_n(z) := (n/2\pi)^{1/2} \int_{\mathbb{R}} e^{-n(t-z)^2/2} T(t)x\,dt \quad (n \in \mathbb{N};\, x \in \mathbb{C}).$$

Prove that the above integrals converge (strongly) in X, $x_n(\cdot)$ is entire (for each $n \in \mathbb{N}$), $x_n \to x$, and $x_n(s) = T(s)x_n$ for all $n \in \mathbb{N}$ and $s \in \mathbb{R}$. Conclude that x_n is an *entire vector* for $T(\cdot)$ (that is, $T(\cdot)x_n$ extends to an entire function), and that consequently *the entire vectors for $T(\cdot)$ are dense in X.*

Nonhomogeneous ACP

10. Let $T(\cdot)$ be a C_o-semigroup, A its generator, and let M, a be constants such that $\|T(t)\| \le M\, e^{at}$ for all $t \ge 0$ (cf. Theorem 1.1). Given a strongly C^1 function $f : [0, \infty) \to X$ and $x \in X$, define

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds \quad (t \ge 0). \tag{31}$$

(a) Prove that u is C^1 on $[0, \infty)$ with values in $D(A)$, and solves the (nonhomogeneous) abstract Cauchy problem

$$(ACP): \quad \frac{du}{dt} = Au + f \quad (t > 0); \quad u(0) = x. \tag{32}$$

(b) *Conversely*, if u is a solution of (ACP) with f strongly continuous, then

$$\frac{d}{ds}T(t-s)u(s) = T(t-s)f(s) \quad (0 \le s \le t), \tag{33}$$

so that u has necessarily the form (31).

In particular, if f is strongly C^1, (ACP) has the unique solution (31). (This is *Duhamel's formula*.)

The Graph Norm on $D(A)$

11. Let A be the generator of the C_o-semigroup $T(\cdot)$ on the Banach space X. Let $[D(A)]$ denote the normed space $D(A)$ normed by the *graph norm* of A

$$\|x\| := \|x\| + \|Ax\| \quad (x \in D(A)). \tag{34}$$

Prove:

(a) $[D(A)]$ is a Banach space.
(b) $S(\cdot) := T(\cdot)|_{D(A)}$ is a C_o-semigroup on $[D(A)]$.
(c) If B is the generator of $S(\cdot)$ (acting in the Banach space $[D(A)]$), then $D(B) = D(A^2)$ and $B = A|_{D(A^2)}$.

Commutativity

12. Let A be the generator of the C_o-semigroup $T(\cdot)$ on the Banach space X, and let $B \in B(X)$. Prove that the following statements are equivalent:

(a) $[B,\, T(\cdot)] = 0$ ($[P, Q]$ denotes here the Lie product $PQ - QP$ of not necessarily bounded operators).
(b) $[B,\, R(\lambda; A)] = 0$ for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$.
(c) $[B,\, R(\lambda; A)] = 0$ for all $\lambda > \omega$.
(d) $B\, D(A) \subset D(A)$ and $[B, A] = 0$ on $D(A)$.
(e) There exists a core $\mathcal{D}$ for A such that $B\,\mathcal{D} \subset D(A)$ and $[B, A] = 0$ on $\mathcal{D}$.

Square of the Generator

13. Let $T(\cdot)$ be a C_o-group of isometries on the Banach space X, and let A be its generator. Define $S(\cdot) : [0, \infty) \to B(X)$ by $S(0) = I$ and

$$S(t) = (2\pi t)^{-1/2} \int_{\mathbb{R}} e^{-s^2/(2t)} T(s)x\, ds \quad (t > 0). \tag{35}$$

Prove:

(a) $S(\cdot)$ is a C_o-semigroup of contractions. Denote its generator by B.
(b) For all $\lambda > 0$,

$$R(\lambda; B) = R(\lambda; (1/2)A^2). \tag{36}$$

Hint: Theorem 1.15 and the identities

$$(2\pi)^{-1/2} \int_0^\infty t^{-1/2} e^{-s^2/(2t)} e^{-\lambda t}\, dt = (2\lambda)^{-1/2} e^{-|s|(2\lambda)^{1/2}} \tag{37}$$

(for all $s \in \mathbb{R}$) and

$$R(\sqrt{2\lambda}; A) + R(\sqrt{2\lambda}; -A) = \sqrt{2\lambda} R(\lambda; (1/2)A^2) \tag{38}$$

(for all $\lambda > 0$). Cf. Theorem 1.11.
(c) Conclude that $B = (1/2)A^2$.
(d) $S(\cdot)$ has an analytic extension in $\mathbb{C}^+$.

Resolvents of Bounded Analytic Semigroups

14. Let A be the generator of the C_o-semigroup $T(\cdot)$, which is analytic in the sector S_θ (cf. Definition 1.53), and bounded in each subsector $S_{\theta-\epsilon}$ $(0 < \epsilon \le \theta)$. Fix such an ϵ, and denote by M_ϵ the supremum of $\|T(\cdot)\|$ over $S_{\theta-\epsilon}$. As in the proof of Theorem 1.54, consider the C_o-semigroups $T_\alpha(\cdot)$ (with generator $A_\alpha := e^{i\alpha}A$) for each real α with $|\alpha| \le \theta - \epsilon$. Prove:

(a) $\sigma(A_\alpha) \subset \{\lambda \in \mathbb{C};\, \Re\lambda \le 0\}$ and $\|R(\lambda; A_\alpha)\| \le M_\epsilon/\Re\lambda$ for $\Re\lambda > 0$.
(b) $\sigma(A) \subset \{\mu \in \mathbb{C};\, |\arg\mu| \ge \theta + \pi/2\}$ and $\|\mu\, R(\mu; A)\| \le M_\epsilon$ for all $\mu \in S_{\theta-\epsilon}$.
(c) $\|\mu\, R(\mu; A)\|$ is bounded in each sector $S_{(\theta+\pi/2)-\epsilon}$.

A-boundedness

15. Let A be the generator of the contraction C_o-semigroup $T(\cdot)$ on the Banach space X, and let $\mathcal{D}$ be a core for A. Let $B : \mathcal{D} \to X$ be linear and satisfy the inequality

$$\|Bx\| \le a\,\|Ax\| + b\,\|x\| \quad (x \in \mathcal{D}) \tag{39}$$

for some constants $a, b \ge 0$. Prove that B extends uniquely as an A-bounded operator with domain $D(A)$ (cf. Definition 1.28). Hint: show that $\mathcal{D}_1 := (I - A)\mathcal{D}$ is dense in X and extend $C := B\,R(1; A)$, originally defined and *bounded* on $\mathcal{D}_1$, (uniquely) to an element of $B(X)$. The wanted extension is $C(I - A)$.

16. Let A, B be (usually unbounded) operators on the Banach space X such that $D(A) \subset D(B)$. Suppose that the (bounded) operator $B\,R(\lambda; A)$ exists and is *compact* for some λ. Prove:

(a) $B\,R(\mu; A)$ is compact for all $\mu \in \rho(A)$.
(b) B is A-bounded.
(c) If X has the *approximation property* (that is, every compact operator on X is the $B(X)$-limit of finite rank operators, e.g., if X is a Hilbert space, cf. [RN, p. 204]), then the A-bound of B is zero (cf. Definition 1.28). Hint: given $\epsilon > 0$, there exist $x_k \in X$ and $x_k^* \in X^*$ ($k = 1, \ldots, n$) such that

$$\left\| B\,R(\lambda; A)x - \sum_k (x_k^* x)x_k \right\| < \epsilon\,\|x\| \quad (x \in X). \tag{40}$$

By density of $R(\lambda; A)^* X^*$ in X^*, we may take $x_k^* = R(\lambda; A)^* y_k^*$ for appropriate $y_k^* \in X^*$, and conclude that

$$\left\| By - \sum_k (y_k^* y)x_k \right\| < \epsilon\,\|(\lambda I - A)y\| \quad (y \in D(A). \tag{41}$$

Unitary Vectors

17. Let $T(\cdot)$ be a contraction C_o-semigroup on the *Hilbert* space X. A *unitary vector* for $T(\cdot)$ is a vector x such that $\|T(t)x\| = \|x\|$ for all $t \geq 0$. Denote by Y the set of all unitary vectors for $T(\cdot)$, and let

$$Z := \bigcap_{t \geq 0} T(t)Y. \tag{42}$$

Prove:

(a) Y is a closed invariant subspace for $T(\cdot)$.
(b) For each $s \geq 0$, $T(s)$ is an isometry of Z onto Z (i.e., $T(\cdot)|_Z$ is a unitary semigroup).
(c) If W is a closed $T(\cdot)$-invariant subspace of $Z^\perp$ such that $T(\cdot)|_W$ is unitary, then $W \subset Z$ and $T(t)W^\perp \subset W^\perp$ (in particular, Z is a reducing subspace for $T(t)$ for all $t \geq 0$, and the only $T(\cdot)$-invariant subspaces of $Z^\perp$ on which $T(\cdot)$ is a unitary semigroup are the trivial ones; one says in this case that $T(\cdot)$ is *completely nonunitary* on $Z^\perp$).

Markov Semigroups

18. Let K be a compact Hausdorff space, and let $C(K)$ be the Banach space of all complex continuous functions on K with the maximum norm. Let $C^+(K)$

be the positive cone in $C(K)$ ($:= \{f \in C(K); f \geq 0\}$). A Markov semigroup is a C_o-semigroup $T(\cdot)$ on $C(K)$ such that $T(\cdot)1 = 1$ and $T(\cdot)f \geq 0$ for all $f \in C^+(K)$. Let A be the generator of the Markov semigroup $T(\cdot)$. Prove:

(a) $\|T(\cdot)\| = 1$.
(b) If $f \in D(A)$, then $\overline{f} \in D(A)$ and $A\overline{f} = \overline{Af}$.
(c) $1 \in D(A)$ and $A1 = 0$.
(d) For all $\lambda > 0$, $\lambda R(\lambda; A)f \geq 0$ for all $f \in C^+(K)$, $\lambda R(\lambda; A)1 = 1$, and $\|\lambda R(\lambda; A)\| = 1$.

Translation Semigroup

19. Denote by $C_l([0,\infty))$ the Banach space of all complex continuous functions f on $[0,\infty)$ for which the limit $f(\infty) := \lim_{s\to\infty} f(s)$ exists in $\mathbb{C}$, with the supremum norm $\|f\| = \sup_{s\geq 0} |f(s)|$. If X is either $C_l([0,\infty))$ or $L^p(\mathbb{R}^+)$ ($1 \leq p < \infty$), consider the translation operators

$$T(t) : f(s) \to f(t+s) \quad (f \in X;\ t \geq 0).$$

Prove:

(a) $T(\cdot)$ is a C_o-semigroup on X with $\|T(\cdot)\| = 1$, and its generator A is the differentiation operator $f \to f'$ with "maximal domain"

$$D(A) = \{f \in X; \exists f',\ f' \in X\}. \tag{43}$$

(In the L^p case, $\exists f'$ means that the derivative f' exists almost everywhere, etc.)

(b) $\sigma(A) = \{\lambda \in \mathbb{C};\ \Re\lambda \leq 0\}$ and $\lambda R(\lambda; A)$ are contractions for all $\lambda > 0$.
(c) Consider the translation operators $T(t)$ ($t \in \mathbb{R}$) on X, where X is either $L^p(\mathbb{R})$ ($1 \leq p < \infty$) or the space $C_l(\mathbb{R})$ of all complex continuous functions f on $\mathbb{R}$ for which both $f(\infty)$ and $f(-\infty)$ exist in $\mathbb{C}$ (with the supremum norm $\|f\| = \sup_{\mathbb{R}} |f|$). Prove that $T(\cdot)$ is a C_o-group of isometries, whose generator A is the differentiation operator with maximal domain in X. Show also that $\sigma(A) = i\mathbb{R}$, and that $\mathcal{S}(\mathbb{R})$ is a core for A.
(d) Let B be any (usually unbounded) operator on a Banach space X with nonempty resolvent set $\rho(B)$. Prove that if λ and $-\lambda$ are both in $\rho(B)$, then $\lambda^2 \in \rho(B^2)$ and

$$R(\lambda^2; B^2) = -R(\lambda; B)\, R(-\lambda; B). \tag{44}$$

(e) If A is the generator of the translation group on $C_l(\mathbb{R})$, then A^2 generates a contraction C_o-semigroup $S(\cdot)$.
(f) Prove that the semigroup $S(\cdot)$ in Part (e) is the Gauss–Weierstrass semigroup (cf. Example 1.114).
(g) For A as in Part (e), prove that A^3 is *not* the generator of a C_o-semigroup.

The MacLaurin Formula for Semigroups

20. Let X be a Banach space, and consider the classical Volterra operator

$$V : u(t) \to \int_0^t u(s)\,ds \quad (t \geq 0), \tag{45}$$

defined on continuous X-valued functions u on $[0, \infty)$.

Let A generate the C_o-semigroup $T(\cdot)$ on X. Prove:

(a) For all $n \in \mathbb{N}$, $t > 0$, and $x \in D(A^n)$,

$$T(t)x = \sum_{k=0}^{n-1} (t^k/k!)A^k x + (V^n[T(\cdot)A^n x])(t). \tag{46}$$

(b) For all $n \in \mathbb{N}$ and $x \in D(A^n)$,

$$\lim_{t \to 0+} \left[T(t)x - \sum_{k=0}^{n-1} (t^k/k!)A^k x \right] = A^n x/n. \tag{47}$$

Restriction of Semigroup to Invariant Subspaces

21. Let A be the generator of the C_o-semigroup $T(\cdot)$ on the Banach space X. Suppose Y is a $T(\cdot)$-invariant linear manifold in X, which is a Banach space for a norm $\|\cdot\|_Y$, such that $(Y, \|\cdot\|_Y)$ is continuously embedded in X. Assume that the semigroup $T_Y(\cdot)$ defined by

$$T_Y(t) := T(t)\Big|_Y \quad (t \geq 0) \tag{48}$$

is of class C_o on the Banach space Y (with the norm $\|\cdot\|_Y$!). Prove that the generator of $T_Y(\cdot)$ is equal to A_Y, *the part of A in Y* (cf. Definition 1.19).

22. (Cf. Exercise 11.) Let A be the generator of the C_o-semigroup $T(\cdot)$, and suppose $0 \in \rho(A)$. For each $n \in \mathbb{N}$, consider the $T(\cdot)$-invariant linear manifold $D(A^n)$ with the norm

$$\|x\|_n := \|A^n x\| \quad (x \in D(A^n)). \tag{49}$$

Prove:

(a) $X_n := (D(A_n), \|\cdot\|_n)$ is a Banach space continuously embedded in X. Denote $T_n(\cdot) := T_{X_n}(\cdot)$ (cf. Exercise 21).

(b) $T_n(\cdot)$ is of class C_o (on X_n). (By Exercise 21, its generator A_n is equal to A_{X_n}, *the part of A in X_n*.

(c) (Write $X_0 := X$, $T_0(\cdot) := T(\cdot)$, and $A_0 := A$.) For all $n = 0, 1, 2, \dots$, A_n is an isometry of X_{n+1} onto X_n, and

$$A_n T_{n+1}(\cdot) = T_n(\cdot) A_n. \tag{50}$$

(d) If X_{-n} are defined inductively for $n = 0, 1, 2, \dots$ such that $X_{-(n+1)}$ is the completion of X_{-n} for the norm

$$\|x\|_{-(n+1)} := \|A_{-n}^{-1} x\|, \tag{51}$$

and $T_{-(n+1)}(t)$ is the continuous extension of $T_{-n}(t)$ to $X_{-(n+1)}$, then Parts (b) and (c) are valid also for all $n \in -\mathbb{N}$.

(e) $D(A_n) = X_{n+1}$ for all $n \in \mathbb{Z}$.

(f) If $m \geq n \in \mathbb{Z}$, then A_n is the (unique) continuous extension of the isometry A_m of X_{m+1} onto X_m to an isometry of X_{n+1} onto X_n.

Semigroups Arising from ACP

23. (Cf. Theorem 1.2.)

Let A be a closed operator on the Banach space X. Consider the associated Abstract Cauchy Problem (ACP) on $[0, \infty)$

$$\frac{du}{dt} = Au; \quad u(0) = x. \tag{ACP}$$

Assume that (ACP) has a unique C^1-solution (in the s.o.t.)

$$u : [0, \infty) \to D(A) \tag{52}$$

for each $x \in D(A)$.

Define $T(\cdot)$ on the Banach space $[D(A)]$ (the Banach space $D(A)$ with the graph norm $\|\cdot\|_A$ of A) by letting

$$T(\cdot)x := u \quad (x \in D(A)), \tag{53}$$

where u is the unique C^1-solution of (ACP). Prove:

(a) For each $t \geq 0$, $T(t)$ is a linear everywhere defined operator on $[D(A)]$, and satisfies the semigroup relations

$$T(s+t) = T(s)T(t) \quad (s.t \geq 0); \quad T(0) = I. \tag{54}$$

(I denotes here the identity operator in $[D(A)]$.)

(b) For each $x \in [D(A)]$, $T(\cdot)x$ is $[D(A)]$-continuous.

(c) The linear operator

$$W : x \in [D(A)] \to T(\cdot)x \in C([0,r],[D(A)]) \tag{55}$$

is a *closed* operator. (The space $C(\cdots)$ denotes the Banach space of all $[D(A)]$-valued $[D(A)]$-continuous functions on $[0,r]$, $r>0$ arbitrary, with the obvious norm.) Hint: use the integral equation equivalent to (ACP). Conclude that $T(t) \in B([D(A)])$ for all $t \geq 0$ (and consequently $T(\cdot)$ is a C_o-semigroup on $[D(A)]$).

(d) Prove that the generator of the C_o-semigroup $T(\cdot)$ (on $[D(A)]$) is *the part of A in $D(A)$*. (Cf. Exercise 11.)

Bounded Below Semigroups

24. A C_o-semigroup $T(\cdot)$ is *bounded below* if $\inf_{t>0} \|T(t)\| > 0$. Prove that $T(\cdot)$ is bounded below iff $\|T(t)\| \geq 1$ for all $t > 0$.

Natural Operational Calculus for Groups

25. Let $T(\cdot)$ be a C_o-group on the Banach space X. Denote by $\mathcal{M}_T$ the space of all complex Borel measures μ on $\mathbb{R}$ such that

$$\|\mu\|_T := \int_{\mathbb{R}} \|T(\cdot)\|\, d|\mu| < \infty. \tag{56}$$

(a) Prove that $\mathcal{M}_T$ is a Banach algebra (with convolution of measures as multiplication) with respect to the norm $\|\cdot\|_T$, and is a Banach subspace of $C_0(\mathbb{R})^*$ if $T(\cdot)$ is bounded below on $(0,\infty)$ (cf. Exercise 24).

(b) Denote the Fourier–Stieltjes transform of the Borel measure μ by $\hat{\mu}$:

$$\hat{\mu}(s) := \int_{\mathbb{R}} e^{ist} d\mu(t) \quad (s \in \mathbb{R}). \tag{57}$$

Let

$$\mathcal{A}_T := \{\hat{\mu};\, \mu \in \mathcal{M}_T\}. \tag{58}$$

Prove that $\mathcal{A}_T$ is a Banach algebra for the pointwise operations between functions, with respect to the norm

$$\|\hat{\mu}\|_T := \|\mu\|_T \quad (\mu \in \mathcal{M}_T). \tag{59}$$

(c) For $f \in \mathcal{A}_T$ (say $f = \hat{\mu}$ for the unique $\mu \in \mathcal{M}_T$), define the operator $\tau(f)$ on X by

$$\tau(f)x := \int_{\mathbb{R}} T(t)x\, d\mu(t) \quad (x \in X). \tag{60}$$

Prove that τ is a continuous homomorphism of the Banach algebra $\mathcal{A}_T$ into $B(X)$ with norm 1, such that

$$\tau(f_t) = T(t) \quad (t \in \mathbb{R}), \tag{61}$$

where $f_t(s) = e^{ist}$, $s, t \in \mathbb{R}$.

(d) If $S(\cdot)$ and $T(\cdot)$ are C_o-groups on X, then $\mathcal{A}_S \subset \mathcal{A}_T$ if and only if $\frac{\|T(\cdot)\|}{\|S(\cdot)\|}$ is bounded on $\mathbb{R}$.

(e) The C_o-group $T(\cdot)$ is *temperate* if $\|T(t)\| = O(|t|^k)$ for some non-negative integer k. In that case, prove that the Schwartz space $\mathcal{S}(\mathbb{R})$ is topologically contained in the Banach algebra $\mathcal{A}_T$, and for all $x \in X$,

$$\tau(f)x = \int_{\mathbb{R}} T(t)x\, \hat{f}(t)\, dt \quad (f \in \mathcal{S}(\mathbb{R})), \tag{62}$$

where $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(t) := (2\pi)^{-1} \int_{\mathbb{R}} e^{-ist} f(s)\, ds \quad (t \in \mathbb{R}).$$

Construction of Analytic Semigroups

26. Let A be a closed densely defined operator on the Banach space X. Suppose there exists $0 < \delta \le \pi/2$ such that the open sector $S_{\delta+\pi/2}$ is contained in $\rho(A)$ and $\lambda R(\lambda; A)$ is bounded in each subsector $S_{\delta-\epsilon+\pi/2}$ with $0 < \epsilon < \delta$.

Fix $\delta' \in (0, \delta)$ and $z \in S_{\delta'}$. Denote $b = 1/|z|$, choose $\epsilon = (\delta - \delta')/2$, set $\alpha = \delta - \epsilon + \pi/2$, and let Γ be the "positively oriented" path consisting of the circular arc

$$\{w = b\, e^{i\theta};\ |\theta| \le \alpha\},$$

complemented by the two rays

$$\{w = r\, e^{\pm i\alpha};\ r \ge b\}.$$

Define

$$T(z) = (1/2\pi i) \int_{\Gamma} e^{zw} R(w; A)\, dw. \tag{63}$$

Prove:

(a) The integral converges in operator norm, and there exists a constant $C_{\delta'}$ such that $\|T(z)\| \le C_{\delta'}$ for all $z \in S_{\delta'}$.

(b) $T(\cdot)$ is analytic in S_δ.

(c) $T(z+u) = T(z)T(u)$ for all $z, u \in S_\delta$. Hint: by Cauchy's integral theorem, the integration path in the definition of $T(u)$ can be translated to the right. Use the resolvent identity and replace the paths by closed paths, using circular arcs as above with radii tending to infinity.

(d) $\lim_{z\in S_{\delta'};\, z\to 0} T(z)x = x$ for all $x \in X$ and $\delta' \in (0,\delta)$. (In particular, the restriction of $T(\cdot)$ to $\mathbb{R}^+$, complemented with the definition $T(0) = I$, is a C_o-semigroup, which admits an analytic extension in S_δ that is bounded in each proper subsector. Denote its generator by B.)

(e) $B = A$. Hint: prove that $R(\lambda; B) = R(\lambda; A)$ for some $\lambda > 0$; cf. Theorem 1.15. (*This construction shows that an operator A satisfying the hypothesis of the present exercise is the generator of an analytic semigroup, whose analytic extension in a suitable sector is bounded in every proper subsector.* The converse is also true.)

Approximation of C_o-semigroups by Uniformly Continuous Semigroups

27. Let $T(\cdot)$ be a C_o-semigroup, and let A be its generator. Suppose B is a *bounded* (everywhere defined) operator *commuting* with $T(\cdot)$. Prove:

(a) For all $x \in D(A)$ and $t \geq 0$, the following identity holds:

$$T(t)x - e^{tB}x = \int_0^t e^{(t-s)B}T(s)(A-B)x\,ds. \tag{64}$$

Hint: integrate with respect to s over $[0,t]$ the expression for

$$\frac{d}{ds}[e^{(t-s)B}T(s)x]$$

when $x \in D(A)$.

(b) If the semigroups $T(\cdot)$ and e^{tB} are bounded (say, by the constants H, K, respectively), then

$$\|T(t)x - e^{tB}x\| \leq HKt\,\|(A-B)x\| \quad (x \in D(A);\, t \geq 0). \tag{65}$$

(c) Suppose $T(\cdot)$ is bounded, B_n, $n \in \mathbb{N}$ are *bounded* operators *commuting* with $T(\cdot)$, and $\|e^{tB_n}\| \leq K$ for all $t \geq 0$ and $n \in \mathbb{N}$ (with a constant K independent of n). Prove that if $B_n x \to Ax$ for all x in a core for A, then

$$T(t) = \lim_n e^{tB_n} \quad (t \geq 0) \tag{66}$$

in the s.o.t., uniformly for t in bounded intervals.

(d) If $T(\cdot)$ is a contraction C_o-semigroup, and

$$B_n := n[T(1/n) - I] \quad (n \in \mathbb{N}), \tag{67}$$

then (66) is valid.

(e) If $T(\cdot)$ is a bounded C_o-semigroup, and B_n are chosen as the Hille–Yosida approximations of A

$$B_n := n[n\,R(n;A) - I] \quad (n \in \mathbb{N}), \tag{68}$$

then (66) is valid. (Cf. Lemma 1.16 and the beginning of the proof of Theorem 1.17.)

Stability in the u.o.t

28. Let ω be the type of the C_o-semigroup $T(\cdot)$. Prove that *the following statements are equivalent*:

(a) $\omega < 0$ (and consequently there exist constants $a > 0$ and $M \geq 1$ such that $\|T(t)\| \leq M\, e^{-at}$ for all $t \geq 0$, that is, $\|T(t)\|$ *decays exponentially to zero* as $t \to \infty$).
(b) $\lim_{t\to\infty} \|T(t)\| = 0$.
(c) $\|T(c)\| < 1$ for some $c > 0$.
(d) $r(T(c)) < 1$ for some $c > 0$.

(Hint: Theorem 1.4.)

29. *Fix* $p \in [1, \infty)$. Let $T(\cdot)$ be a C_o-semigroup on the Banach space X such that

$$\|T(\cdot)x\| \in L^p(\mathbb{R}^+) \tag{69}$$

for all $x \in X$. (The norm on $L^p(\mathbb{R}^+)$ will be denoted by $\|\cdot\|_p$.) Fix $a > 0$ and $M \geq 1$ such that $\|T(t)\| \leq M\, e^{at}$ for all $t \geq 0$ (cf. Theorem 1.1). Define the map

$$V : X \to L^p(\mathbb{R}^+)$$

by

$$V(x) = \|T(\cdot)x\| \quad (x \in X). \tag{70}$$

Prove:

(a) There exists a constant $K > 0$ such that $\|V(x)\|_p \leq K\, \|x\|$ for all $x \in X$.
(b) For all $t > 0$,

$$\|T(t)\| \leq (ap)^{1/p} MK. \tag{71}$$

(Hint: integrate with respect to s over $[0, t]$ the trivial identity

$$e^{-aps}\|T(t)x\|^p = e^{-aps}\|T(s)T(t-s)x\|^p$$

and estimate the integral on the right-hand side.)

(c) For all $t > 0$,

$$\|T(t)\| \leq (ap)^{1/p} M\, K^2 t^{-1/p}. \tag{72}$$

Hint: integrate (with respect to s over $[0, t]$) the trivial identity

$$\|T(t)x\|^p = \|T(t-s)T(s)x\|^p,$$

and estimate the integral on the right-hand side.)

(d) Conclude that the equivalent relations in Exercise 28 are also equivalent to the property (69) for some (*or all*) $p \in [1, \infty)$.

Semigroups on Hilbert Space

30. (This exercise is a preliminary for the next one.)

Let X be a (complex) Hilbert space. Denote by $C_c^\infty(\mathbb{R}, X)$, $\mathcal{S}(\mathbb{R}, X)$, $L^2(\mathbb{R}, X)$, etc. the "usual" spaces C_c^∞, $\mathcal{S}$, L^2, etc. (respectively) of X-valued functions u over $\mathbb{R}$, with definitions adequately modified so that the norm $\|u(\cdot)\|$ replaces the usual absolute values. Fix an orthonormal basis $\{x_j;\, j \in J\}$ for X. Prove:

(a) $u \in L^2(\mathbb{R}; X)$ if and only if $(u(\cdot), x_j) \in L^2(\mathbb{R})$ for all $j \in J$. When this is the case, one has the identity

$$\|u\|^2_{L^2(\mathbb{R},X)} = \sum_{j \in J} \|(u(\cdot), x_j)\|^2_{L^2(\mathbb{R})}. \tag{73}$$

(b) If $u \in L^2(\mathbb{R}, X)$ and $\phi \in L^1(\mathbb{R})$, then the convolution $u * \phi$ is a "well-defined" element of $L^2(\mathbb{R}, X)$, and

$$\|u * \phi\|_{L^2(\mathbb{R},X)} \le \|u\|_{L^2(\mathbb{R},X)} \|\phi\|_{L^1(\mathbb{R})}. \tag{74}$$

(c) $C_c^\infty(\mathbb{R}, X)$ (and therefore $\mathcal{S}(\mathbb{R}, X)$) is dense in $L^2(\mathbb{R}, X)$. (Cf. proof of Theorem II.1.2 in [K17].)

(d) Define the Fourier transform $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}, X)$ by

$$(\mathcal{F}u)(s) = (1/\sqrt{2\pi}) \int_{\mathbb{R}} e^{-ist} u(t)\, dt. \tag{75}$$

(In the special case $X = \mathbb{C}$, it is well-known that $\mathcal{F}$ is a linear isometry of $\mathcal{S}(\mathbb{R})$ onto itself with respect to the $L^2(\mathbb{R})$ norm on $\mathcal{S}(\mathbb{R})$; cf. for example [K17, p. 379].) Prove that, for any Hilbert space X, $\mathcal{F}$ is a (linear) *isometry* of $\mathcal{S}(\mathbb{R}, X)$ onto itself with respect to the $L^2(\mathbb{R}, X)$ norm, that is,

$$\int_{\mathbb{R}} \|\mathcal{F}u\|^2\, ds = \int_{\mathbb{R}} \|u\|^2\, dt \tag{76}$$

for all $u \in \mathcal{S}(\mathbb{R}, X)$, and conclude that $\mathcal{F}$ *extends uniquely as a linear isometry of* $L^2(\mathbb{R}, X)$ *onto itself.*

Stability in the u.o.t. on Hilbert Space

31. (This exercise is related to Exercises 28 and 29.) Let $T(\cdot)$ be a C_o-semigroup on the *Hilbert* space X, let A be its generator and ω its type. Fix $M \ge 1$ such that $\|T(t)\| \le M\, e^{(|\omega|+1)t}$ for all $t \ge 0$ (cf. Theorem 1.1) and $a > |\omega| + 1$. Define $S(\cdot) : \mathbb{R} \to B(X)$ by setting $S(t) = 0$ for $t < 0$ and

$$S(t) = e^{-at} T(t) \quad (t \ge 0). \tag{77}$$

Prove:

(a) For each $x \in X$, one has $S(\cdot)x \in L^2(\mathbb{R}, X)$ and

$$R(a + i\cdot; A)x = \sqrt{2\pi}\mathcal{F}[S(\cdot)x]. \tag{78}$$

(b) For each $x \in X$,

$$\int_{\mathbb{R}} \|R(a + is; A)x\|^2 \, ds = 2\pi \int_0^\infty \|S(t)x\|^2 \, dt. \tag{79}$$

(Cf. Exercise 30.)

(c) There exists a constant $K > 0$ such that

$$\|R(a + i\cdot)x\|_{L^2(\mathbb{R},X)} \leq K \|x\| \quad (x \in X). \tag{80}$$

(d) *Assume in the sequel that* $R(\cdot; A)$ *exists and is (uniformly) bounded in* $\mathbb{C}^+$. Observe that the imaginary axis is then contained in $\rho(A)$ (cf. Theorem 1.11). (Therefore the resolvent is uniformly bounded in $\overline{\mathbb{C}^+}$; denote $L := \sup_{\Re z \geq 0} \|R(z; A)\|$). Prove that for all $x \in X$ and $s \in \mathbb{R}$

$$\|R(is; A)x\| \leq (1 + aL) \|R(a + is)x\|. \tag{81}$$

(e) Prove that for all $x \in X$

$$\|R(i\cdot; A)x\|_{L^2(\mathbb{R},X)} \leq (1 + aL)K \|x\|, \tag{82}$$

and similarly (by considering the adjoint semigroup, cf. Exercise 4),

$$\|R(i\cdot; A^*)x\|_{L^2(\mathbb{R},X)} \leq (1 + aL)K \|x\|. \tag{83}$$

(f) Apply Theorem 1.15(3) and integration by parts to prove that for all $x \in X$ and $t > 0$,

$$2\pi i \, tT(t)x = \int_{\mathbb{R}} e^{(a+is)t} R(a + is; A)^2 x \, ds. \tag{84}$$

(g) Prove the inequality

$$2\pi|(tT(t)x, y)| \leq \|R(i\cdot; A)x\|_{L^2(\mathbb{R},X)} \|R(-i\cdot; A^*)y\|_{L^2(\mathbb{R},X)} \tag{85}$$

for all $x, y \in X$, and conclude that

$$\|T(t)\| \leq H/t \quad (t > 0), \tag{86}$$

with $H = (1 + aL)^2 K^2/(2\pi)$. In particular, $\lim_{t\to\infty} \|T(t)\| = 0$, that is, *the assumption in Part (d) implies that* $T(\cdot)$ *satisfies the equivalent properties in Exercise 28*. Conversely, if $\omega < 0$, $R(\lambda; A)$ exists and is uniformly bounded in $\mathbb{C}^+$ (cf. Theorem 1.11), that is, the latter property is *equivalent* to the properties in Exercise 28 (when X is a Hilbert space).

Hille–Yosida Space, Semi-Simplicity Space, etc.

32. (The following exercises give a unified construction of the Hille–Yosida space, the semi-simplicity space, the Laplace–Stieltjes space, etc.)

Let X be a Banach space, and let $\mathcal{A}$ be a subset of $B(X)$ containing the identity I in its strong closure. Denote

$$\|x\|_{\mathcal{A}} := \sup_{T\in\mathcal{A}} \|Tx\| \quad (x \in X) \tag{87}$$

and

$$Z = Z(\mathcal{A}); = \{x \in X;\ \|x\|_{\mathcal{A}} < \infty\}. \tag{88}$$

Prove:

(a) Z with the norm $\|\cdot\|_{\mathcal{A}}$ is a Banach subspace of X.
(b) For any S in the commutant $\mathcal{A}'$ of $\mathcal{A}$ in $B(X)$, $SZ \subset Z$ and $S|_Z \in B(Z)$ with $\|S|_Z\|_{B(Z)} \leq \|S\|$.
(c) If $\mathcal{A}$ is a semigroup (for operator multiplication), then Z is $\mathcal{A}$-invariant, and $\mathcal{A}|_Z$ ($:= \{T|_Z;\ T \in \mathcal{A}\}$) is contained in the closed unit ball $B_1(Z)$ of $B(Z)$.
(d) (Hypothesis as in Part (c).) If W is an $\mathcal{A}$-invariant Banach subspace of X such that $\mathcal{A}|_W \subset B_1(W)$, then W is a Banach subspace of Z. (This is the *maximality* of Z with respect to the property in Part (c).)

33. Let A be an operator with domain $D(A)$ in the Banach space X, whose resolvent set contains a ray (a,∞) ($a \in \mathbb{R}$). Let $\mathcal{A}$ be the unital semigroup generated by the operator family

$$(\lambda - a)R(\lambda; A) \quad (\lambda > a). \tag{89}$$

Denote by A_Z the *part of A in* $Z = Z(\mathcal{A})$, and let W be *the closure of* $D(A_Z)$ *in the Banach subspace* Z (cf. Exercise 32). Prove:

(a) A_W generates a C_o-semigroup $T(\cdot)$ on W such that $\|T(t)\|_{B(W)} \leq e^{at}$.
(b) If V is a "resolvent-invariant" (i.e., $R(\lambda; A)V \subset V$ for all $\lambda > a$) Banach subspace of X such that A_V generates a C_o-semigroup $S(\cdot)$ on V such that $\|S(t)\|_{B(V)} \leq e^{at}$, then V is a Banach subspace of W and $S(\cdot) = T(\cdot)|_V$.
(Cf. Definition 1.22 and Theorem 1.23; the Banach subspace W is precisely the Hille–Yosida space for A, denoted there by Z. Part (b) is the "maximality" property of the Hille–Yosida space. Cf. Theorem 1.17.)

34. Let A be a (generally unbounded) operator on the Banach space X, whose resolvent set contains the (open) sector S_θ, for some $\theta \in (0, \pi/2]$. Let $\mathcal{A}$ be the unital semigroup of $B(X)$ generated by the operator family

$$\{\lambda R(\lambda; A);\ \lambda \in S_\theta\}. \tag{90}$$

As in Exercise 33, denote by A_Z the part of A in $Z = Z(\mathcal{A})$, and let W be the closure of $D(A_Z)$ in the Banach subspace Z. Prove:

(a) A_W generates an analytic C_o-semigroup of contractions on W in the sector S_θ.

(b) If V is a resolvent-invariant Banach subspace of X such that A_V generates an analytic C_o-semigroup of contractions on V in the sector S_θ, then V is a Banach subspace of W.

(The Banach subspace W is the *analytic Hille–Yosida space for* A; Part (b) states its "maximality." Cf. Corollary 3 in Section C of Part I.)

35. Let $T(\cdot)$ be a holomorphic C_o-semigroup on $\mathbb{C}^+$, acting in the Banach space X. Let

$$\mathcal{A} = \{T(\lambda);\ \lambda \in \mathbb{C}^+\}. \tag{91}$$

($\mathcal{A}$ is a semigroup whose strong closure contains the identity, by the C_o condition.) Let $Z = Z(\mathcal{A})$ ($T(\cdot)$-invariant!), and define

$$Z_b := \{x \in Z;\ \lim_{t\to 0+} \|[T(t) - I]x\|_{\mathcal{A}} = 0\}. \tag{92}$$

Prove:

(a) Z_b is a $T(\cdot)$-invariant Banach subspace of X, and $T(\cdot)|_{Z_b}$ is a holomorphic contraction C_o-semigroup on $\mathbb{C}^+$ in the Banach space Z_b. (Therefore $T(\cdot)|_{Z_b}$ possesses a *boundary group*, which is a C_o-group of isometries on Z_b. Cf. Theorem 1.105.)

(b) Z_b is a "maximal" Banach subspace of X with the property in Part (a).

36. Let K be a compact nowhere dense set in $\mathbb{C}$ with connected complement. Let T be a bounded operator on the Banach space X, with spectrum contained in K. As usual, $C(K)$ denotes the Banach algebra of all complex continuous functions f on K with the norm $\|f\|_K := \sup_K |f|$. By Lavrentiev's theorem, the subalgebra $\mathcal{P}(K)$ of polynomials (restricted to K) is dense in $C(K)$ (cf. [Ga, Theorem 8.7]). Let

$$\mathcal{A} = \{p(T);\ p \in \mathcal{P}(K), \|p\|_K \le 1\}, \tag{93}$$

and let $Z = Z(\mathcal{A})$ be the associated Banach subspace (cf. Exercise 32). Prove:

(a) There exists a unique continuous representation $\tau : C(K) \to B(Z)$ such that $\|\tau(f)\|_{B(Z)} \le \|f\|_K$ for all $f \in C(K)$ and $\tau(p) = p(T)$ for all $p \in \mathcal{P}(K)$.

(b) If W is a T-invariant Banach subspace of X with the property in Part (a) (with W replacing Z), then W is a Banach subspace of Z.

(c) If X is reflexive, there exists a unique spectral measure E on Z supported by K, which commutes with every $S \in B(X)$ commuting with T, such that $\|E(\cdot)\|_{B(Z)} \le 1$ and

$$\tau(f)x = \int_K f(\lambda)\, E(d\lambda)x \quad (x \in Z;\ f \in C(K)). \tag{94}$$

(The Banach subspace Z in this case is the semi-simplicity space for T.)

37. (The preceding exercise does not apply for example to the case $K = \Gamma := \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$. The present exercise indicates how to modify the construction in this case.)

Let T be a bounded operator on the Banach space X with spectrum on the unit circle Γ.

Let $\mathcal{R}(\Gamma)$ denote the algebra of restrictions to Γ of all complex polynomials in λ and λ^{-1} ($= \overline{\lambda}$ on Γ).

If $p \in \mathcal{R}(\Gamma)$, we may write

$$p(\lambda) = \sum_{k\in\mathbb{Z}} \alpha_k \lambda^k, \tag{95}$$

where $\alpha_k \in \mathbb{C}$ and only finitely many α_k are nonzero. We then define

$$p(T) = \sum_{k\in\mathbb{Z}} \alpha_k T^k. \tag{96}$$

(Note that T is invertible, since $\sigma(T) \subset \Gamma$!) Set

$$\mathcal{A} := \{p(T);\ p \in \mathcal{R}(\Gamma),\ \|p\|_\Gamma \le 1\}, \tag{97}$$

and let $Z = Z(\mathcal{A})$ be the associated Banach subspace of X. Prove:

(a) There exists a continuous representation $\tau : C(\Gamma) \to B(Z)$ such that $\tau(p) = p(T)$ for all $p \in \mathcal{R}(\Gamma)$ and $\|\tau(f)\|_{B(Z)} \le \|f\|_\Gamma$ for all $f \in C(\Gamma)$.
(b) Z is "maximal" in the "usual sense" with respect to the property in Part (a) (cf. preceding exercise).
(c) If X is reflexive, Part (c) of Exercise 36 is valid (with $K = \Gamma$).

38. (This "exercise" is merely a remark on Exercise 32 and Section B of Part II.) Let X be a Banach space, and let

$$F : [0,\infty) \to B(X)$$

be a strongly continuous function such that $F(0) = I$. Choose

$$\mathcal{A} = \left\{ \int_0^\infty \phi(s) F(s)\, ds;\ \phi \in C_c^\infty(\mathbb{R}^+),\ \|\phi\|_\infty = 1 \right\}. \tag{98}$$

The Banach subspace $Z := Z(\mathcal{A})$ in the present case is the Laplace–Stieltjes space for F (cf. Section B in Part II). When F is a contraction C_o-semigroup $T(\cdot)$, $\mathcal{A}$ is a semigroup, and if X is reflexive, Z coincides topologically with the semi-simplicity space for the generator $-A$ of $T(\cdot)$ as defined in Section A of Part II (when $\mathbb{R}^+ \subset \rho(-A)$).

39. Let $T(\cdot)$ be a C_o-group of operators on the Banach space X, and denote its generator by iA. Let

$$\mathcal{A} = \left\{ \sum_k \alpha_k T(t_k);\ \alpha_k \in \mathbb{C},\, t_k \in \mathbb{R},\, \left| \sum_k \alpha_k e^{it_k s} \right| \le 1 \right\}, \tag{99}$$

where the sums in (99) are finite and $s \in \mathbb{R}$.

(In this case, $\mathcal{A}$ is a unital semigroup, so that Exercise 32 applies in all its parts.) If X is reflexive, prove that the Banach subspace $Z = Z(\mathcal{A})$ coincides with the semi-simplicity space for $T(\cdot)$. (Cf. Section B, Part I.)

40. (Cf. Section B in Part II and Chapter 10 in [17].) Let A be an unbounded operator with *real* spectrum on the Banach space X, and denote by T its Cayley transform

$$T := (iI - A)(iI + A)^{-1} = -2iR(-i; A) - I. \tag{100}$$

Since $\sigma(T)$ lies on the unit circle Γ (cf. [DS I–III, Lemma VII.9.2]), we may consider the Banach subspace $Z = Z(\mathcal{A})$ associated with the semigroup $\mathcal{A}$ defined in (97). Prove:

(a) There exists a continuous representation

$$\tau : C(\overline{\mathbb{R}}) \to B(Z)$$

such that

$$\|\tau(f)\|_{B(Z)} \le \|f\|_\infty \quad (f \in C(\overline{\mathbb{R}}))$$

and $\tau(\phi) = T|_Z$ for $\phi(s) = \frac{i-s}{i+s}$.

(b) If X is reflexive, then there exists a spectral measure on Z

$$F : \mathcal{B}(\mathbb{R}) \to B(Z),$$

such that $\|F(\cdot)\|_{B(Z)} \le 1$, F commutes with every $U \in B(X)$ which commutes with A, and

(i) $D(A_Z)$ is the set of all $x \in Z$ such that the integral

$$\int_{\mathbb{R}} s\, F(ds)x := \lim_{a \to -\infty,\, b \to \infty} \int_a^b s\, F(ds)x$$

exists in X and belongs to Z;

(ii) for all $x \in D(A_Z)$,

$$Ax = \int_{\mathbb{R}} s\, F(ds)x,$$

and

(iii) for all nonreal $\lambda \in \mathbb{C}$ and $x \in Z$,

$$R(\lambda; A)x = \int_{\mathbb{R}} \frac{1}{\lambda - s} F(ds)x.$$

(c) Formulate (and prove) a "maximality and uniqueness" statement for the Banach subspace Z with respect to the properties in Parts (a) and (b).

(d) Let $P(\cdot,\cdot)$ be the *Poissonian* of A

$$P(t,s) := \frac{1}{2\pi i}[R(t-is;A) - R(t+is;A)] \quad (t \in \mathbb{R};\ s > 0).$$

Consider the operators

$$U_s : C_c(\mathbb{R}) \to B(X) \quad (s > 0)$$

defined by

$$U_s h = \int_{\mathbb{R}} h(t)P(t,s)\,dt \quad (s > 0;\ h \in C_c(\mathbb{R})).$$

Let

$$\mathcal{A}_1 := \{I\} \cup \{U_s h;\ s > 0,\ h \in C_c(\mathbb{R}),\ \|h\|_\infty \le 1\},$$

and let $Z_1 := Z(\mathcal{A}_1)$ be the associated Banach subspace as in Exercise 32. Set

$$Z' := \{x \in Z_1;\ \lim_{|u|\to\infty} R(\cdot + iu;A)x = 0\}.$$

For X reflexive, prove that Z' is a closed subspace of Z_1 which coincides with Z, with equality of norms.

Approximation Formula for the Integrated Semigroup

41. Let $T(\cdot)$ be a *bounded* C_o-semigroup on the Banach space X, and let A be its generator. Since $T(\cdot)x$ is continuous on $[0,\infty)$ for each $x \in X$ (by the C_o-condition, cf. Theorem 1.1), the X-valued function $f\,T(\cdot)x$ is Bochner integrable for each $f \in L^1(\mathbb{R}^+)$ and $x \in X$, and we may then define a map

$$\tau : L^1(\mathbb{R}^+) \to B(X)$$

by

$$\tau(f)x = \int_0^\infty f(s)T(s)x\,dt \quad (f \in L^1(\mathbb{R}^+);\ x \in X). \tag{101}$$

(Clearly $\|\tau\| \le M := \sup\|T(\cdot)\|$.)

(a) Let $f_{n,t}(s) := 1 - \exp\Big(-e^{n(t-s)}\Big)$ $(t > 0)$. Prove that $f_{n,t} \to \chi_{[0,t]}$ in $L^1(\mathbb{R}^+)$ as $n \to \infty$, uniformly in t on bounded intervals in $\mathbb{R}^+$ ($\chi_{[0,t]}$ denotes the characteristic function of $[0,t]$). Consequently,

$$\tau(f_{n,t})x \to \int_0^t T(s)x\,ds \quad (x \in X;\ t > 0) \tag{102}$$

as $n \to \infty$, strongly in X and uniformly in t on bounded intervals.

(b) Prove that

$$\int_0^t T(s)x\,ds = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} e^{tkn} R(kn;A)x \quad (x \in X;\, t > 0), \tag{103}$$

strongly in X and uniformly in t on bounded intervals in $\mathbb{R}^+$.

Semigroup Induced on Quotient Space

42. Let $T(\cdot)$ be a C_o-semigroup on the Banach space X, and let A be its generator. Suppose Y is a closed $T(\cdot)$-invariant subspace, and let $\pi : X \to X/Y$ be the canonical map. Define $V(t) \in B(X/Y)$ by

$$V(t)(\pi x) = \pi(T(t)x) \quad (t \geq 0;\, x \in X). \tag{104}$$

Prove:

(a) $V(\cdot)$ is a C_o-semigroup on X/Y.
(b) If B denotes the generator of $V(\cdot)$, then

$$D(B) = \pi(D(A))$$

and

$$B(\pi x) = \pi(Ax) \quad (x \in D(A)). \tag{105}$$

Semigroup Induced on $l^\infty(X)$

43. Let $T(\cdot)$ be a C_o-semigroup on the Banach space X, and let A be its generator. Let

$$Y = l^\infty(X) := \{y := \{y_n\};\ y_n \in X,\ \|y\|_Y := \sup_n \|y_n\| < \infty\}. \tag{106}$$

For each $t \geq 0$, define

$$S(t)y := \{T(t)y_n\} \quad (y \in Y). \tag{107}$$

Set

$$W := \{y \in Y;\ \lim_{t\to 0+} \|S(t)y - y\|_Y = 0\}. \tag{108}$$

Prove:

(a) $S(\cdot)$ is a C_o-semigroup on the Banach space W.
(b) If B denotes the generator of $S(\cdot)$, then

$$D(B) = \{y \in W;\ y_n \in D(A),\ \textit{and}\ \{Ay_n\} \in W\ \textit{for all}\ n\}$$

and

$$By = \{Ay_n\} \quad (y \in D(B)). \tag{109}$$

Semigroup Induced on a Tensor Space

44. Let X, Y be Banach spaces, and let $X \otimes Y$ be their algebraic tensor product (its typical element is a finite sum $\sum x_k \otimes y_k$, with $x_k \in X$, $y_k \in Y$). If $\|\cdot\|$ is a *cross norm* on $X \otimes Y$ (that is, $\|x \otimes y\| = \|x\|\,\|y\|$ for all $x \in X$, $y \in Y$), denote by $X \hat{\otimes} Y$ the completion of $X \otimes Y$ under the given cross norm. Let $T(\cdot),\ S(\cdot)$ be C_o-semigroups on X, Y (respectively), with respective generators A, B. For each $t \geq 0$, define

$$V(t) = T(t) \otimes S(t),$$

where the right-hand side is defined as the unique continuous extension of the operator defined on $X \otimes Y$ by

$$V(t)\left(\sum x_k \otimes y_k\right) = \sum (T(t)x_k) \otimes (S(t)y_k).$$

Prove:

(a) $V(\cdot)$ is a C_o-semigroup on $X \times Y$, and $\|V(t)\| = \|T(t)\|\,\|S(t)\|$ for all $t \geq 0$.

(b) The (algebraic) tensor product $D(A) \otimes D(B)$ is a core for the generator C of $V(\cdot)$, and C is equal to the closure of the operator

$$A \otimes I + I \otimes B$$

defined on $D(A) \otimes D(B)$.

Infinite Product of Semigroups

45. Let $\{T_k(\cdot)\}$ be a sequence of pairwise commuting contraction C_o-semigroups on the Banach space X. Let A_k be the generator of $T_k(\cdot)$, $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, set

$$P_n(\cdot) := \prod_{k=1}^{n} T_k(\cdot).$$

Prove:

(a) For each $n \in \mathbb{N}$, $P_n(\cdot)$ is a C_o-semigroup, and its generator is the closure of the operator $\sum_{k=1}^{n} A_k$ with domain $\bigcap_{k=1}^{n} D(A_k)$.

(b) A *convergence vector* for the given sequence is a vector

$$x \in \bigcap_{k=1}^{\infty} D(A_k)$$

such that

$$\sum_{k=1}^{\infty} \|A_k x\| < \infty. \tag{110}$$

Assume that the subspace D of all convergence vectors for the given sequence is dense in X. Prove that for each $t \geq 0$, $P_n(t)$ converges strongly to an operator $P(t) \in B(X)$, uniformly with respect to t in compact intervals, and $P(\cdot)$ is a contraction C_o-semigroup, called the *infinite product* of the given semigroups (denoted $\prod_{k=1}^{\infty} T_k(\cdot)$).

(c) The subspace D of convergence vectors is a core for the generator A of $P(\cdot)$, and $Ax = \sum_{k=1}^{\infty} A_k x$ for all $x \in D$.
Hints: for Part (b), observe that

$$\|P_{n+m}(t)x - P_n(t)x\| \leq \left\| \prod_{k=n+1}^{n+m} T_k(t)x - x \right\|$$

and

$$\left\| \prod_{k=n+1}^{p} T_k(t)x - \prod_{k=n+1}^{p-1} T_k(t)x \right\| \leq \|T_p(t)x - x\|$$

for all $p \geq n+1$, with an "empty product" equal to I by definition.
For Part (c), note that D is $T(\cdot)$-invariant and dense in X (by hypothesis).

Perturbation of Generator by $B \in B([D(A)])$

46. Let $T(\cdot)$ be a C_o-semigroup on the Banach space X, and let A be its generator. Denote by $[D(A)]$ the Banach space defined by the domain of A with the graph norm. Let $B \in B([D(A)])$ and fix $\lambda_0 \in \rho(A)$. Since

$$P := (\lambda_0 I - A)B\,R(\lambda_0; A) \in B(X) \tag{111}$$

(why?), we may choose $\lambda > \omega$ (the type of $T(\cdot)$!) so that

$$\|PR(\lambda; A)\| < 1. \tag{112}$$

Define

$$Q := (\lambda_0 I - A)B\,R(\lambda; A)\ (\in B(X)), \tag{113}$$

$$R := (\lambda I - A)B\,R(\lambda; A)\ (\in B(X)), \tag{114}$$

and

$$U := I - B\,R(\lambda; A). \tag{115}$$

(a) Prove that U is invertible in $B(X)$, and $U\,D(A) = D(A)$.
Hint: in any Banach algebra, the sets of nonzero elements of $\sigma(ab)$ and $\sigma(ba)$ coincide (cf. for example [K17, p. 173]). In the Banach algebra $B(X)$, take $a = Q$ and $b = R(\lambda_0; A)$, so that $ba = B\,R(\lambda; A)$ and $ab = P\,R(\lambda; A)$.

(b) Prove that

$$U(A+R)U^{-1} = A+B.$$

(c) Conclude that $A+B$ (with domain $D(A)$) generates a C_o-semigroup on X.

Intertwining and Spectrum

47. Let $T(\cdot)$ and $S(\cdot)$ be C_o-semigroups on the Banach spaces X and Y, respectively, and let $A,\ B$ be their respective generators. Suppose $C \in B(Y,X)$ intertwines the given semigroups, that is,

$$CS(\cdot) = T(\cdot)C.$$

Prove:

(a) $CB \subset AC$.
(b) $CR(\lambda;B) = R(\lambda;A)C$ for all $\lambda \in \rho(A) \cap \rho(B)$.
(c) If CY is dense in X, then $CD(B)$ is a core for A.
(d) Suppose $0 \in \rho(B)$, and let then Ω be an open disk with radius $r > 0$ centered at 0 such that $\overline{\Omega} \subset \rho(B)$. Suppose also that $R(\lambda;A)$ exists and $\|R(\lambda;A)\| \le |\Re\lambda|^{-1}$ for $\Re\lambda \neq 0$. Finally, suppose that CY is dense in X. Fix $x \in X$ and a sequence $\{y_n\} \subset Y$ such that $Cy_n \to x$. Define

$$y_n(\lambda) := [1+(\lambda/r)^2]CR(\lambda;B)y_n \quad (\lambda \in \overline{\Omega}).$$

Prove

$$\|y_n(re^{i\theta})\| \le (2/r)\|Cy_n\| \quad (\theta \in [0,2\pi),\ \cos\theta \neq 0),$$

and conclude that

$$\|y_n(\lambda)\| \le (2/r)\|Cy_n\| \quad (\lambda \in \Omega),$$

and therefore

$$\|R(\lambda;A)x\| \le \frac{2r}{|r^2+\lambda^2|}\|x\| \quad (\lambda \in \Omega,\ \Re\lambda \neq 0),$$

hence

$$\|R(\lambda;A)\| \le (8/3r) \quad (|\lambda| < r/2,\ \Re\lambda \neq 0).$$

Consequently, $0 \in \rho(A)$. Conclude that

$$\sigma(A) \subset \sigma(B) \cap i\mathbb{R}.$$

Mining Lemma 2.16

48. Let (S, Σ, σ) be a σ-finite positive measure space, and let Ω be a locally compact Hausdorff space. Denote by $M(\Omega)$ the space of all regular complex Borel measures on Ω. Let

$$T : L^1(S, \Sigma, \sigma) \to C_0(\Omega)$$

be a bounded linear operator. Prove:

(a) Given $\phi \in L^\infty(S, \Sigma, \sigma)$ and a constant $K > 0$, there exists $\mu \in M(\Omega)$ such that

$$\phi = T^*\mu; \quad \|\mu\| \le K \tag{116}$$

if and only if

$$\left| \int_S f\phi \, d\sigma \right| \le K \, \|Tf\|_\infty \tag{117}$$

for all f in a dense subset of $L^1(S, \Sigma, \sigma)$. (Cf. Lemma 2.16.)

(b) Suppose $\phi_n = T^*\mu_n$ with $\mu_n \in M(\Omega)$, $\|\mu_n\| \le K$, $n = 1, 2, \ldots$, and $\phi_n \to \phi$ pointwise almost everywhere on S. Then $\phi = T^*\mu$ for some $\mu \in M(\Omega)$ with $\|\mu\| \le K$. If T^* is injective, then $\mu = w^* - \lim \mu_n$. (This may be interpreted as a general version of the Paul Levy "continuity theorem.")

(c) A sequence $\phi = \{\phi_n\} \in l^\infty$ is the moments sequence of a measure $\mu \in M([0,1])$ with $\|\mu\| \le K$ if and only if

$$\left| \sum c_n \phi_n \right| \le K \max_{[0,1]} \left| \sum c_n t^n \right| \tag{118}$$

for all finite complex vectors $(c_1, \ldots, c_n)$. (Hint: choose $T : l^1 \to C([0,1])$ in an appropriate way.) State an analogous criterion for the trigonometric moments problem.

(d) A bounded continuous function ϕ on $\mathbb{R}$ is the cosine-Stieltjes transform

$$\phi(t) = \int_{\mathbb{R}} \cos(st)\mu(ds) \quad (t \in \mathbb{R})$$

of a measure $\mu \in M(\mathbb{R})$ with $\|\mu\| \le K$ if and only if

$$\left| \int_{\mathbb{R}} f\phi \, dt \right| \le K \|\tilde{f}\|_\infty, \tag{119}$$

where $\tilde{f}$ denotes the *cosine transform* of f

$$\tilde{f}(s) := \int_{\mathbb{R}} \cos(st) f(t) \, dt. \tag{120}$$

(e) Let $k \in C_b(\mathbb{R})$ have a convergent improper Riemann integral on $\mathbb{R}$. Given $\phi \in C_b(\mathbb{R})$ and $K > 0$, one has

$$\phi(t) = (k * \mu)(t) := \int_{\mathbb{R}} k(t-s)\mu(ds) \quad (t \in \mathbb{R}) \tag{121}$$

for some $\mu \in M(\mathbb{R})$ with $\|\mu\| \le K$ if and only if

$$\left| \int_{\mathbb{R}} f\phi \, dt \right| \le K \, \|\check{k} * f\|_\infty \tag{122}$$

for all f in a dense subset of $L^1(\mathbb{R})$.
(We used the following notation: $\check{k}(t) := k(-t)$; if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then $f * g$ is their convolution.)
Classical kernels k for which Part (e) applies are the *Dirichlet kernel* $k(t) = (\sin t)/t$, the *Fejer kernel* $k(t) = [(\sin t)/t]^2$, and the *Poisson kernel*

$$k_\epsilon(t) = \frac{\epsilon}{\pi(\epsilon^2 + t^2)}, \tag{123}$$

where $\epsilon > 0$. In the latter case, prove the *uniqueness* of the representation of ϕ, when it exists.

49. Let A be a commutative (complex) Banach algebra, and let $\mathcal{M}$ be its *structure space* (that is, the space of all regular maximal ideals with the Gelfand topology). Denote by $G : A \to C_0(\mathcal{M})$ the Gelfand transform ($Gx = \hat{x}$). Given $x^* \in A^*$ and a constant $K > 0$, prove that there exists a measure $\mu \in M(\mathcal{M})$ such that

$$x^* = G^*\mu; \quad \|\mu\| \le K \tag{124}$$

if and only if

$$|x^*x| \le K \, r(x) \tag{125}$$

for all x in a dense subset of A. ($r(x)$ denotes the spectral radius of x. Cf. Lemma 2.16.)

The Eberlein and Schoenberg Criteria for Fourier–Stieltjes Transforms

50. Let G be a locally compact Abelian group. Denote by $L^1(G)$ its *group algebra* (that is, its L^1 space with respect to the Haar measure dt on G, with convolution as multiplication). Let Γ be the dual group $\hat{G}$. Given a bounded continuous function ϕ on G and a constant $K > 0$, prove that ϕ is the Fourier–Stieltjes transform

$$\phi(t) = \int_{\Gamma} (t, \gamma)\mu(d\gamma) \quad (t \in G) \tag{126}$$

of some $\mu \in M(\Gamma)$ with $\|\mu\| \le K$ if and only if

$$\left| \int_G f\phi \, dt \right| \le K \, \|\hat{f}\|_\infty \tag{127}$$

for all f in a dense subset of $L^1(G)$.

Cf. Exercise 49; (t, γ) denotes the value of the character $\gamma \in \Gamma$ at the point $t \in G$, and $\hat{f}$ denotes the Fourier transform of $f \in L^1(G)$, that is,

$$\hat{f}(\gamma) := \int_G (t, \gamma) f(t) \, dt \quad (\gamma \in \Gamma). \tag{128}$$

(The above statement is known as *the Eberlein criterion* for Fourier–Stieltjes transforms on locally compact Abelian groups; the special case for $G = \mathbb{R}$ is the *Schoenberg criterion.*)

51. Let X, Y be Banach spaces, and $T \in B(X, Y)$. Prove that the restriction of T^* to the (strongly) closed unit ball of Y^* has *weak**-closed range. (Cf. Lemma 2.16.)

52. Suppose iS generates a C_o-group $S(\cdot) : \mathbb{R} \to B(X)$ on the Banach space X, and $V \in B(X)$ leaves $D(S)$ invariant and $[S, V] \subset V^2$. Let $T_a(\cdot)$ be the C_o-group generated by iT_a, where $T_a = (S - V) + V_a$ and $V_a = S(a) V \, S(-a)$, $a \in \mathbb{R}$ (cf. Corollary of Theorem 1.38). Prove:

(a) For all $a, t \in \mathbb{R}$,

$$T_a(t) = S(t) - atV \, S(t) V_a.$$

(b) For all $a, t \in \mathbb{R}$ such that $t \ne -a$,

$$T_a(t) = S(t) + i \frac{at}{a + t} [S(a + t), V] \, S(-a).$$

(Cf. Theorem 3.18 and its proof.)

Notes and References

The theory of operator semigroups was essentially started by the 1932 paper of M. H. Stone on groups of unitary operators in Hilbert space [St]. The general Banach space theory was established in the following two decades, and is detailed in the classical 1957 monograph [HP]. Later results are included and/or referred to in many more recent books, some of which are cited in our bibliography.

Part I. General Theory

Standard books on the general theory of operator semigroups are [D3, EN1, EN2, G, HP, P]. Chapters on the subject are also included in general texts on Functional Analysis such as [DS I–III, Kat1, RS, Y].

A. Basic Theory

Most of the material concerning the interplay between a semigroup and its generator, culminating with the Hille–Yosida theorem, was developed in the 1930s and 1940s, and is necessarily found in any text on operator semigroups.

We comment below on some more recent results included in this section.

The Hille–Yosida space. The terminology and Theorem 1.23 are from [K5].

The Trotter–Kato convergence theorem. Theorem 1.32 goes back to [Tr].

Exponential formulas. The treatment here follows [D3, P], and is based on work by [Kat3, Ch1, Ch2, Tr].

Perturbations of generators. Theorem 1.38 is due to Hille–Phillips. The proof given here is essentially the one in [DS I–III].

Groups of operators. Theorem 1.40 is from [N]. Theorem 1.41 is the classical Stone theorem [St].

B. The Semi-simplicity Space for Groups

Theorem 1.49 and the following analysis are from [K3].

C. Analyticity

Theorem 1.54 is a variation on a result of [Liu]. The proof given here is from [K9].

D. The Semigroup as a Function of its Generator

Noncommutative Taylor formula. The results of this section are from [K7].
Analytic families of semigroups. The results are from [K10].

E. Large Parameter

The results of this section are (in the exposition order) from [K12, K13, K14, K15], [KP], and [AB]. See also [LV].

F. Boundary Values

The main facts about "regular semigroups" are contained in [HP, Theorems 17.9.1 and 17.9.2]; the "converse" part (and the corollaries) are from [K16].

G. Pre-Semigroups

The concept appears in germinal form in [DaP] (under the name of "regularizable semigroups"). In [DP], the name "C-semigroup" is coined, and the detailed analysis of these families is started (see [DL1, DL2, DL3, M1, M2, MT1, MT2, MT3, T1, T2], as a partial list for this subject). Since a C-semigroup is *not* a semigroup (unless $C = I$), we call it here a *pre-semigroup*.
Theorems 1.119–1.121 are from [DL1].
Theorem 1.124 is from [DL2] (but we coined the term *exponentially tamed* as a reference to Property 3).

Part II. Integral Representations

A. The Semi-Simplicity Space

The concept goes back to [K1] for a single *bounded* operator, with extensions to unbounded operators appearing in [K1, K2, KH2, KH3]. Theorem 2.3 is from [KH2]. A variant of this theorem is found in [KH3]. Theorem 2.12 is from [K2] (see also [K4]). Lemma 2.16 is from [KH3] (see also [DLK]).

B. The Laplace–Stieltjes Space

The concepts of the *Laplace–Stieltjes space* and of the *Integrated Laplace space* for a family of closed operators were introduced and studied in [DLK]. Theorems 2.15, 2.20, 2.21, and 2.23 are from [DLK] (with some modification of the proofs). Theorem 2.28 is a special case of the main result of [DL3]. *Integrated semigroups* were introduced in [Neu].

C. Families of Unbounded Symmetric Operators

Semigroups of unbounded symmetric operators. First results on this subject were obtained in [De] and [Nus]. A general theory of semigroups of unbounded operators in Banach space was developed in [H1, H2]. Theorem 2.29 is from [KL], as well as the proof of Theorem 2.31 (the latter theorem appeared originally in [Nel], with a different proof). Another proof of Theorem 2.29 is found in [Fr], and serves as a model for the proofs of Theorems 2.35 and 2.37 (first published in [KH3]) for local cosine families of symmetric operators. Frohlich's proof was modified in [V1] to generalize the Frohlich–Klein–Landau theorem to local symmetric semigroups defined on *semigroups* of $\mathbb{R}^+$. For local symmetric semigroups defined on general topological semigroups, see [V2]. The results on local semigroups are generalized to a Banach space setting in [KH1] (see also [K4]).
Local cosine families of unbounded symmetric operators. The results are from [KH3]. The concept of *semi-analytic vectors* is due to Nussbaum, as well as Theorem 2.39 (with a proof independent of the result on local cosine families of symmetric operators; see [RS]).

Part III. A Taste of Applications

A. Dependence on Parameters

This section is based on [KM].

B. Similarity (etc.)

The results are from [K18, K19, K20], [K23], and [KPe], with some modifications. See also [K4], [KH5], and [VK] for related results and generalizations.

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(The bibliography lists mostly works which are related somehow to the material of the book or which are needed as references for results used in proofs, and which appeared after 1975. For papers published before 1975, we refer to the vast bibliography of [G].)

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